6.849: Geometric Folding Algorithms, Spring 2017

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Problem Set 8 Solutions

Due: Wednesday, April 12, 2017

Solve Problem 8.1 and optionally Problem 8.2.

Problem 8.1 [Mandatory, Collaboration exactly with your project group].

Each week from now on, we will ask you to tell us about your progress from the last week on your final project. What have you been working on or thinking about? Did you run into any issues or questions? Did you reach any milestones? Did your project shift direction? (If you don't have progress from the last week, say so to get credit for this problem, but glance nervously at the impending deadline.)

The following problem is optional. Time permitting, we will still provide feedback and solutions to any submissions.

Problem 8.2 [Optional, Collaboration OK].

Consider three distinct points A, B, C on the surface of a convex polyhedron, and imagine the possible shortest paths p from A to B and shortest paths q from A to C. Prove that at least one of the following cases holds:

(a) There's a shortest path p from A to B, and a shortest path q from A to C, that intersect only at A.

(b) For every such pair of shortest paths, p contains q.

(c) For every such pair of shortest paths, q contains p.

Solution:

It suffices to prove that one of the following two cases holds:

(a) There's a shortest path p from A to B, and a shortest path q from A to C, that intersect only at A.

(b) For every such pair of shortest paths, p contains q or q contains p.

All three original conditions imply one of the new ones, and the first new condition implies the first old one. If the second new condition holds and the shortest distance from A to B is shorter than the shortest distance from A to C, then no path p contains any path q, so the third original condition holds. Similarly, if the shortest distance from A to B is longer than the shortest from A to C, then no path q contains any path p, so the second original condition holds. Finally, if the shortest distances are the same and there's any pair of shortest paths with one containing the other, then their endpoints are the same, that is, $B = C$, contradicting the distinctness of the three points.

Suppose that the first case doesn't hold, that is, for every shortest path p from A to B and shortest path q from A to C, there's an intersection other than at A. Take one such pair of paths p_0 and q_0 with neither containing the other, and consider their intersection X farthest from A, which doesn't equal A by assumption.

Case 1: The portion $p_0 = q_0$ of p between A and X equals the portion of q between A and X. Since neither path contains the other, both paths extend past X, say as p_1 and q_1. Then consider the smallest angle at X between p_0 and p_1, call it p_0Xp_1, and the smallest angle at X between q_0 and q_1, call it q_0Xq_1. Both of them are at most π: in a convex polyhedron, there's at most 2π of angle at every point, so the smaller of the two angular arcs is at most π, with equality possible only if there's 2π of angle at X (i.e. X is flat). If either of these angles is smaller than π, then the corresponding path is not a locally shortest path: there's a shortcut at X. If both of them equal π, then X is flat and both paths are straight lines, that is, X isn't the farthest intersection of the paths, contradicting the definition of X.

Case 2: The portion p_0 of p between A and X doesn't equal the portion q_0 of q between A and X. Let X' be the closest point to A such that the entire segment XX' is shared between the paths. Then $X' \neq A$ by this case's assumption, so there's some segment of p_1 starting at X and extending toward A along p that's distinct from some segment of q_1 starting at X and extending toward A along q. Also, B and C are distinct, so X' is distinct from at least one of them, say B, so there's a portion p_2 of path starting at X and continuing along p toward P. Then consider the smallest angle at X between p_1 and p_2, call it p_1Xp_2, and the smallest angle at X between q_1 and p_2, call it q_1Xp_2. Both of them are at most π: in a convex polyhedron, there's at most 2π of angle at every point, so the smaller of the two angular arcs is at most π, with equality possible only if there's 2π of angle at X (i.e. X is flat). If $p_1Xp_2 < \pi$, then p isn't a shortest path: there's a shortcut at X. If $q_1Xp_2 < \pi$, then p isn't a shortest path, because the path p' consisting of the portion of q from A to X' followed by the portion of p between X' and B has the same length as p, but there's a shortcut to it at X. If both of them equal π, then X is flat and both paths are straight lines, that is, $p_1 = q_1$, contradicting the choice of X' as the closest intersection of the paths to A.