

Gluing results: [Demaine, Demaine, Lubiw, O'Rourke 2002]

(algorithms & combinatorics)

| | # gluings & enumeration alg. | decision algorithm |
|---------------------------|---------------------------------|-----------------------|
| general | $2^{\Theta(n)}$ | $2^{\Theta(n)}$ * |
| edge-to-edge gluing | $2^{\Theta(n)}$ | $n^{O(1)}$ |
| bounded sharpness polygon | $n^{\Theta(1)}$ | $n^{O(1)}$ |

lower bounds on worst-case # gluings (\Rightarrow output size)

* **OPEN**: general polynomial-time decision algorithm
(does the polygon have any gluing?)

Combinatorial type of gluing =

- ① abstract gluing tree (without lengths)
- ② specification of which polygon vertices & edges come together at each gluing-tree junction, including leaves & all points with ≥ 1 polygon vertex

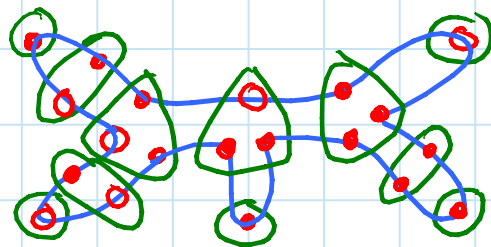
— count just combinatorially distinct gluings
 \Rightarrow remove infinities

Edge-to-edge gluing = only vertices at tree junctions
= only glue whole polygon edges to whole polygon edges

Bounded-sharpness polygon = all angles $\leq 360^\circ - \epsilon$,
(e.g., convex polygons) for some $\epsilon > 0$ indep. of n

$2^{O(n)}$ upper bound on gluings: (combinatorially distinct)

- each leaf in gluing tree is a vertex or fold point
 - $\Rightarrow \leq n+4$ leaves
 - $\Rightarrow O(n)$ tree junctions of degree $\neq 2$
- $\leq n$ tree junctions with ≥ 1 polygon vertex (which includes all degree-2 junctions)
 - $\Rightarrow O(n)$ tree junctions
 - $\Rightarrow 2^{O(n)}$ possible abstract trees
 - $2^{O(n)}$ choices of where each tree junction has polygon vertices (yes/no for each)

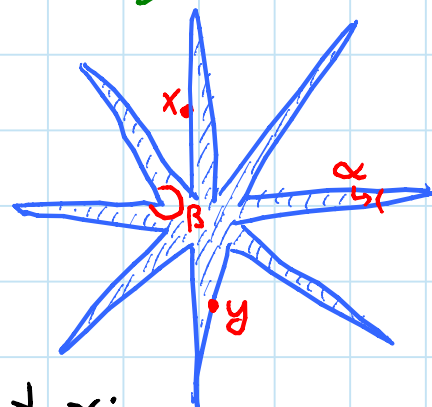


Σ gluing tree
○ junction
• polygon vertex
○ not

- n choices for which is first polygon vertex
 - $\Rightarrow 2^{O(n)} \cdot 2^{O(n)} \cdot n = 2^{O(n)}$ comb. distinct gluings

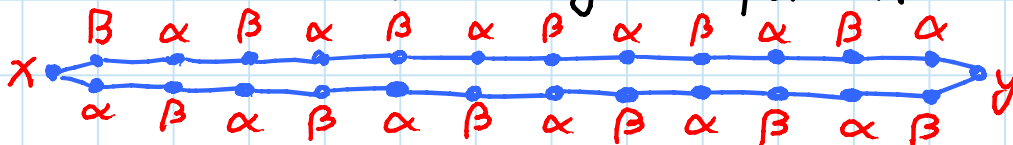
$2^{-\Omega(n)}$ lower bound on gluings: (even edge-to-edge)

- sharpening: $\alpha \rightarrow \varepsilon$
 $\& \beta \rightarrow 360^\circ - \frac{360^\circ}{n/2} - \varepsilon'$
 $\ll 360^\circ - n\varepsilon$

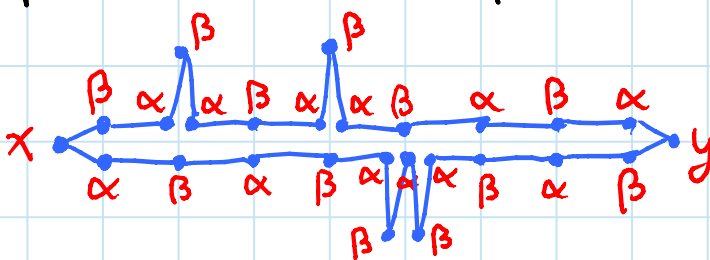


\Rightarrow can glue any number of α 's into a β (but not $> 1 \beta$)

- perimeter halve from edge midpoint x :



- zip $n/4 \beta$'s on top & $n/4 \beta$'s on bottom:



- always an Alexandrov gluing
 $\approx 2^{n/4} \cdot 2^{n/4} = 2^{n/2}$ gluings

Also: $2^{-\Omega(L)}$ gluings with L leaves

[2010 open problem session: Abel, Demaine, Demaine, Eisenstat, Ellowitz, Ku, Lynch, Qi, Morgan, Schardl]

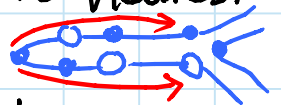
$n^{O(1)}$ upper bound for bounded sharpness: $O(1)$, not $O(1/n)$

- suppose every vertex has angle $\leq 360^\circ - \varepsilon$
- \Rightarrow every leaf in gluing tree has curvature $\geq \varepsilon$
(assuming $\varepsilon \leq 180^\circ$ to allow fold points)
- 720° total curvature
- $\Rightarrow L \leq 720^\circ / \varepsilon = O(1/\varepsilon)$ leaves
- e.g. $L \leq 4$ if polygon convex

$n^{O(L)}$ upper bound for L leaves:

- $< L$ internal junctions of degree ≥ 3
- $2^{O(L)}$ abstract trees on these junctions + leaves
i.e. on all but degree-2 junctions
- $\binom{O(n)}{O(L)}$ assignments of polygon vertices & edges to these tree junctions

- for vertex leaf, can measure distances to nearest branching junction (which has a vertex)



\Rightarrow determine this part of gluing: induct

- trouble: fold point (edge) at a leaf

- especially with rolling belt

- $O(n^2)$ combinatorial types in rolling belt

$O(n^2)$ events during rolling

- $n^{O(1)}$ over ≤ 3 belts

$\Rightarrow \binom{O(n)}{O(L)} \leq n^{O(L)}$ & $2^{O(n)}$ total

- in fact: $O(n^{2L-2})$, $O(n^4)$ for $L=4$, $O(n^2)$ for $L \leq 3$

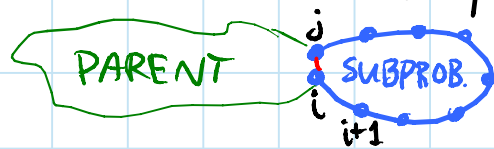


tight for $L=2$

OPEN: better bounds for $L > 2$? $2^{O(L)}$ $n^{O(1)}$? see prev. page

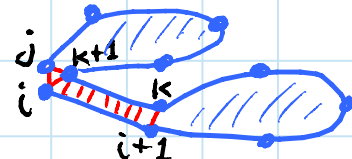
Dynamic program for edge-to-edge gluing: [Lubiw & O'Rourke 1996]

- subproblem = chain $v_i \dots v_j$ of the polygon whose endpoints are glued together



- to which edge does (ccw) first edge $v_i v_{i+1}$ of chain glue?

- at most n choices $v_k v_{k+1}$:
(try them all)



- get two smaller subproblems: $v_{i+1} \dots v_k$ & $v_{k+1} \dots v_j$
 - to list all foldings:
 - combine all pairs of solutions to subproblems
 - discard any with negative curvature
 - exponential, but optimal in worst case
 - decision problem in polynomial time:
 - each subproblem just reports one gluing, with min. possible total angle glued at v_i & v_j
 - take min over all (Alex.) combined options
 - ∞ if no Alexandrov gluing
 - if any gluing allows parent to be Alexandrov, this will be one
 - $O(n^2)$ subproblems, $O(n)$ time each
- $\Rightarrow O(n^3)$ time

Dynamic program for general gluing: [Lubiw 2000; Hirata 2000]

- two types of subproblems:
 - chain $v_i \cdots v_j$ with v_i glued to v_j
 - chain $v_i \cdots e_j$ with v_i glued to point along e_j
 - (- chain $e_i \cdots v_j$ symmetrically)
- answer to subproblem = list of options, each:
 - combinatorial type of gluing
 - total angle of material glued at v_i
 - in $v_i \cdots e_j$ case, interval $\subseteq (0, |e_j|)$ of points along e_j to which v_i can glue in this option
- to solve $v_i \cdots e_j$ (say), there are two cases:

① zip: glue interval after v_i to interval before e_j point

- stop at first vertex, either:

① $v_{i+1} \Rightarrow$ recurse on $v_{i+1} \cdots e_j$

- v_i 's interval = v_{i+1} 's, offset by $+|e_i|$, clipped at $|e_j|$

② $v_j \Rightarrow$ recurse on $e_i \cdots v_j$

- v_i 's interval = v_j 's, cropped at $|e_j|$

③ both v_{i+1} & $v_j \Rightarrow$ recurse on $v_{i+1} \cdots v_j$

② tug: glue another vertex v_k against v_i

- assume v_k is smallest such

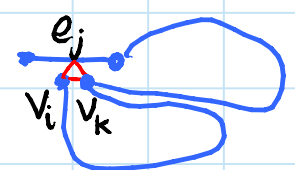
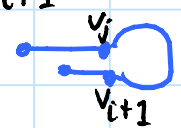
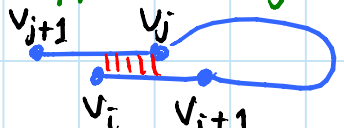
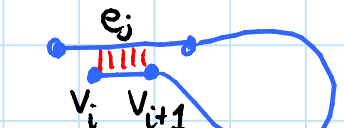
\Rightarrow recurse on $v_i \cdots v_k$ with initial zip

- recurse on $v_k \cdots e_j$ (zip or tug) - v_i 's interval = v_k 's

- in $v_i \cdots v_j$ subproblem, tug can glue edge e_k against v_i & v_j

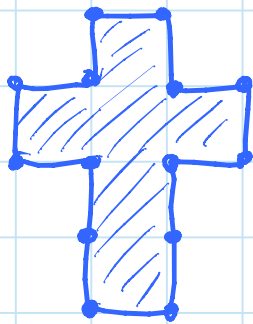
- time = $O(n^3) \cdot \#$ gluings at any stage

$$= \begin{cases} 2^{O(n)} & \text{in general} \\ n^{O(1)} & \text{for bounded sharpness} \end{cases}$$



Case studies:

Latin cross:



[Demaine, Demaine,
Lubiw, O'Rourke]

- 5 edge-to-edge gluings
- 85 general gluings = $42 \cdot 2 + 1$ → symmetric
- 23 different polyhedra:
 - cube
 - 2 flat quads
 - 7 tetrahedra
 - 3 pentahedra
 - 4 hexahedra
 - 6 octahedra

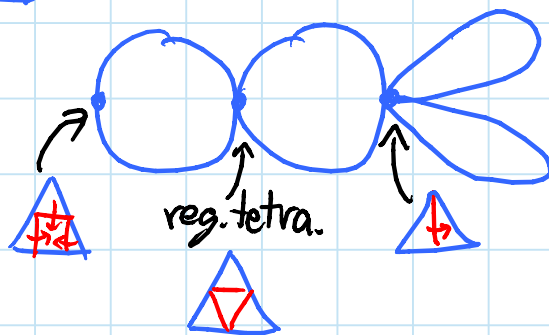
Hirata's Half-Length Theorem: [Hirata 2000]

if edge lengths are integers & no rolling belts
then suffices to cut each edge at midpoint
& use edge-to-edge gluings

Equilateral triangle:



- 2 flat polyhedra
- 4 continua:
 - hexahedra
 - pentahedra
 - tetrahedra











Square:  [Alexander, Dyson, O'Rourke 2003]

- continua of 5 combinatorial types:

- tetrahedra
- 2 pentahedra
- hexahedra
- octahedra

- 4 flat polyhedra:

- | | | | |
|------------|---|---|---|
| - right |  | ⇒ |  |
| - square |  | ⇒ |  |
| - rect. |  | ⇒ |  |
| - pentagon |  | ⇒ |  |

