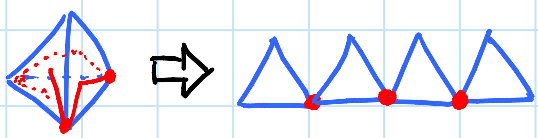
| 6.849 | Lecture 14 | Oct. 27, 2010 |
|---|---|---|

**Recall:**

| | edge unfolding | general unfolding |
|---|---|---|
| convex polyhedra | OPEN | ALWAYS |
| nonconvex polyhedra | NOT ALWAYS | OPEN |

**Vertex unfolding:** [Demaine, Eppstein, Erickson, Hart, O'Rourke 2003]
- different relaxation of edge unfolding
- still cut only along polyhedron edges (all!)
- require connectivity just through vertices (vs. edges)
  (like hinged dissection)

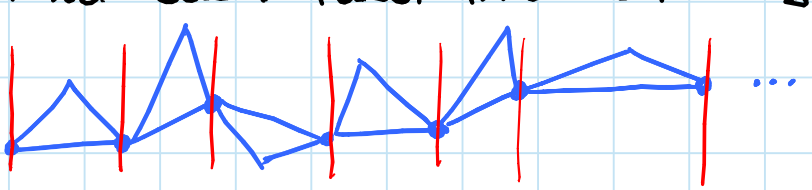
**Example:**



**Universality:** every connected triangulated manifold has a vertex unfolding, computable in linear time

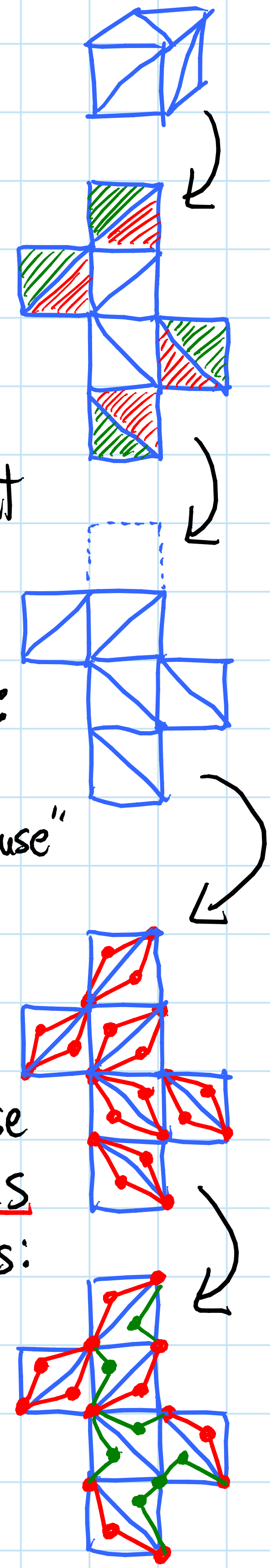
① construct facet path going from facet to facet along vertex adjacencies, visiting every facet exactly once, not repeating a vertex twice in a row
[next page; also Bartholdi & Goldsman 2004]
↳ but $O(n^2)$ time

② unfold each facet into vertical slab:



⇒ no overlap

# Vertex unfolding: (cont'd)



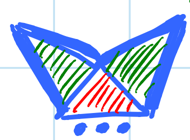
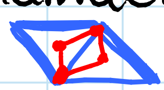
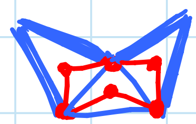
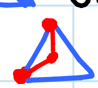
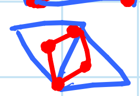
Constructing a facet path for 2D surfaces:

① Cut edges until facets are connected in tree-like fashion
- only removes connections
  $\Rightarrow$ only harder to find path
- duplicates vertices but can repeat

$\rightarrow$ triangulated "polygon" (may self-overlap)

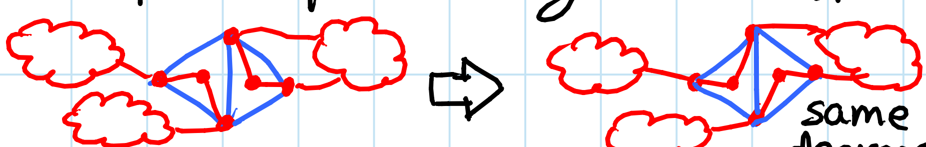
② Color "ears" = $\Delta$s with one adjacent triangle & two boundary edges

③ Color ears in what remains

④ Remove second ear & 1 or 2 first ear(s):
- either  "dunce cap" or  "Mickey Mouse"
- recurse on remainder
- put back with  or 
- base case: nothing or  or 

$\Rightarrow$ every vertex but 2 from base case have even number of connections

⑤ Connect components by local switches:



same degrees

⑥ (Noncrossing) Eulerian path

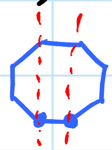
bonus: get cycle $\Leftrightarrow$ not "checkered": 2-color $\Delta$s, only red $\Delta$s on boundary

# Vertex unfolding: (cont'd)

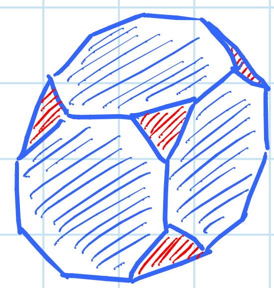
General vertex unfolding is trivial: triangulate

$\boxed{\text{OPEN}}$: does every convex polyhedron have a vertex unfolding?

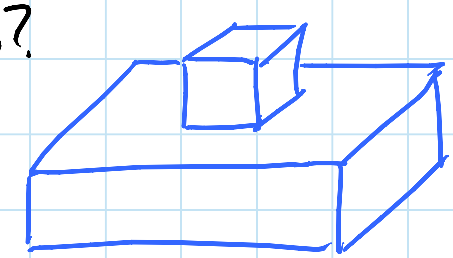
- facet path $\not\Rightarrow$ vertex unfolding:

  no slabs  repeat 3 times

- facet path may not exist:
  - truncated cube has 8 disjoint △s & 6 ⯃s
  - not enough ⯃s to put between △s

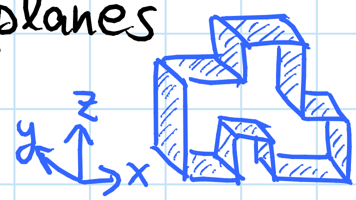


$\boxed{\text{OPEN}}$: does every polyhedron with holeless faces have a vertex unfolding?

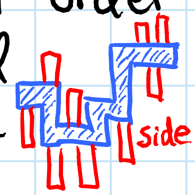
cf.: 

**Orthogonal polyhedra:** (all faces perpendicular to coord. axis)
generally unfoldable if genus $\emptyset$
[Damian, Flatland, O'Rourke 2007]

**Proof:** slice through every vertex with y-plane (xz)
- y-faces encompassed = side faces
- x & z-faces (yz & xy) = band faces
- band faces between two y-planes form a cycle = band

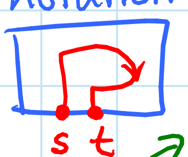


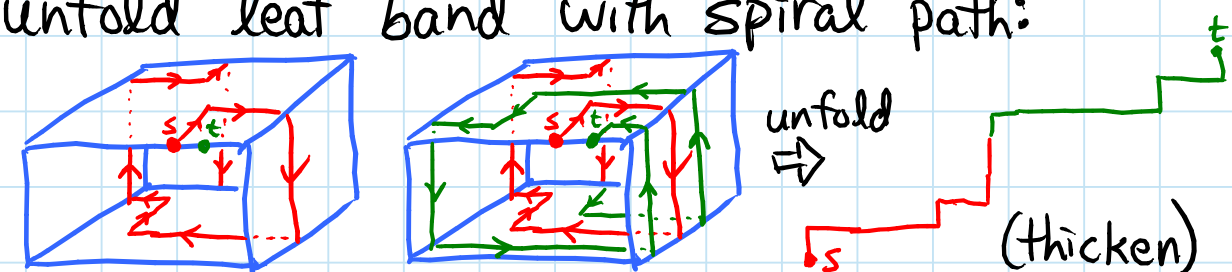

- bands form a tree structure, connecting parent to child with thin side faces
- idea: visit bands in roughly depth-first order
  - unfold to proceed always rightward
  - side faces just attach above/below




- unfold leaf band with spiral path:

notation:

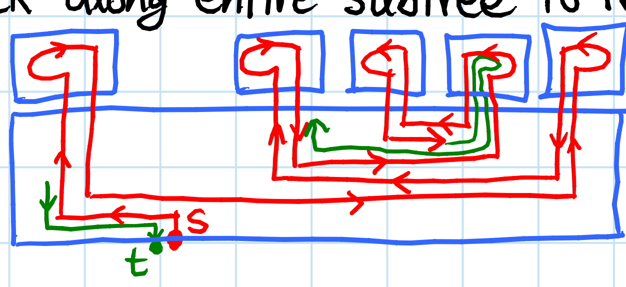



- visit −y children with alternating path:


fewer turns (4 vs. 6) possible [Cheung 2007]

- similarly visit +y children
- double-back along entire subtree to return to parent @ t

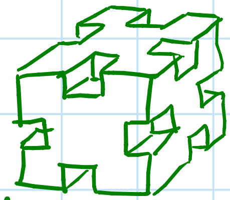


# Orthogonal polyhedra: (cont'd)

**Grid** = slice through every vertex with x,y,z-planes

**Grid unfolding** = just cut along grid

$\boxed{\text{OPEN}}$: does every orthogonal polyhedron have a grid unfolding?
- analog of edge unfolding



edge unfolding impossible

**Refinement** = divide each grid rectangle into $\underbrace{k \times k}$

goal: minimize k

**Summary:**

| | type | $k \times k$ | ref |
|---|---|---|---|
| – general | | $2^{O(n)} \times 2^{O(n)}$ | [DFO 2007] (prev. page) |
| – general | vertex-unf. | $1 \times 1$ | [DFO 2006] |
| – Manhattan towers | | $5 \times 4$ | [DFO 2005] |

↳ connected y=0 base, y-plane slices shrink as y increases

| | type | $k \times k$ | ref |
|---|---|---|---|
| – orthoterrain (rect. base) | | $1 \times 1$ | [O'Rourke 2007] |
| – orthostacks | | $2 \times 1$ | [Biedl et al. 1998] |

↳ every y-plane slice is connected

| | type | $k \times k$ | ref |
|---|---|---|---|
| – orthostacks | vertex-unf. | $1 \times 1$ | [Demaine, Iacono, Langerman 2006] |
| – orthoconvex orthostacks | | $1 \times 1$ | [Damian & Meijer 2004] |

↳ y-plane slices are orthogonally convex: x & y slices connected

| | type | $k \times k$ | ref |
|---|---|---|---|
| – orthotubes | | $1 \times 1$ | [Biedl et al. 1998] |

↳ unit cubes connected face-to-face in open/closed chain

| | type | $k \times k$ | ref |
|---|---|---|---|
| – well-separated orthotrees | | $1 \times 1$ | [DFMeijerO 2005] |

↳ unit cubes connected face-to-face in a tree

↳ no two branching cubes are adjacent

# Orthogonal polyhedra: (cont'd)

$\boxed{\text{OPEN}}$: $n^{O(1)} \times n^{O(1)}$ refinement in general?
$\boxed{\text{OPEN}}$: $\Omega(n) \times \Omega(n)$ lower bound?

$\boxed{\text{OPEN}}$: power of $O(1) \times O(1)$ refinement?
— e.g. orthotrees
$\boxed{\text{OPEN}}$: power of $1 \times 1$ refinement?
— e.g. orthostacks, Manhattan towers

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$\boxed{\text{OPEN}}$: orthogonal polyhedra of higher genus?

$\boxed{\text{OPEN}}$: general unfolding of arbitrary polyhedra of genus $\emptyset$?

$\boxed{\text{PROJECT}}$: implement $2^{O(n)} \times 2^{O(n)}$ method for unfolding orthogonal polyhedra of genus $\emptyset$

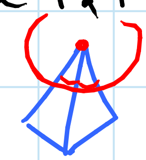
# Cauchy's Rigidity Theorem: [Cauchy 1813; Steinitz 1934]

if two convex polyhedra are combinatorially equivalent
same graph/incidence structure

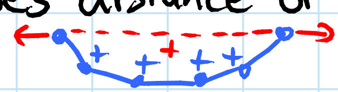
& corresp. faces are congruent, then polyhedra are congruent

**Proof** by contradiction: consider counterexample $P, P'$

- for each vertex pair $v, v'$:
  slice polyhedron with $\varepsilon$-sphere at $v, v'$
  $\Rightarrow$ spherical polygon, edge lengths = face angles,
  angles = dihedral angles

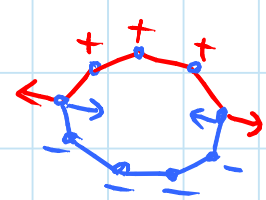


- faces congruent $\Rightarrow$ edge lengths match in $v$ vs. $v'$
- $P, P'$ incongruent $\Rightarrow$ angles don't match for some $v, v'$
- label vertex in $v$'s spherical polygon + if larger angle in $v$,
  $\emptyset$ if equal angles, and − if smaller angle in $v$
- can't be all + (& $\emptyset$) or all − (& $\emptyset$) by:
  Cauchy Arm Lemma: opening all angles of a convex open chain increases distance of endpts. (in plane & sphere)



  $\Rightarrow$ edge length would not be preserved

$\Rightarrow$ $\geq 2$ alternations + + ⋯ + − − ⋯ − + + +



- can't be just 2 by Cauchy Arm Lemma:

$\Rightarrow$ $\geq 4$ alternations + − + − +

$\Rightarrow$ # alternations $\geq 4V$ in subgraph of +/− edges

## Proof of Cauchy's Rigidity Theorem: (cont'd)

- +/− labels extend to entire edge of polyhedron
- +/− alternation at vertex corresponds one-to-one +/− alternation in incident face
- $\le 2k$ alternations in face of $2k$ or $2k+1$ edges

$\Rightarrow 4V \le$ #alternations $\le 2f_3 + 4f_4 + 4f_5 + 6f_6 + 7f_7 + \cdots$

- $E = \frac{1}{2}(3f_3 + 4f_4 + 5f_5 + \cdots)$ (handshaking)
- $V = 2 + E - F$ Euler's Theorem

$= 2 + \frac{1}{2}(f_3 + 2f_4 + 3f_5 + \cdots)$

$\Rightarrow 4V = 8 + 2f_3 + 4f_4 + 6f_5 + \cdots \longrightarrow$ contradiction (+8)

$\square$

## Uniqueness of folding: [Alexandrov 1941]

- suppose you glue polygon's boundary to itself
- what convex polyhedra can it form?
- every edge will be a shortest path
- draw all shortest paths between vertices (points of nonzero curvature)

$\Rightarrow$ fix combinatorial & facial structure

- Cauchy's Rigidity Theorem $\Rightarrow \le 1$ convex realization

NEXT LECTURE: when is there $\ge 1$ convex realization?