6.849	Lecture 11	Oct. 18. 2010

Locked linkages: recall

	chains	[L10]	trees	
2D	never locked	✓	can lock	TODAY
3D	can lock		can lock	
$4D^+$	never locked		never locked	

Algorithms for unfolding 2D chains

① ordinary differential equation given by (canonical) expansive infinitesimal motion $\frac{dE}{dt} = \underline{d}$

[Connelly, Demaine, Rote 2000, 2002]

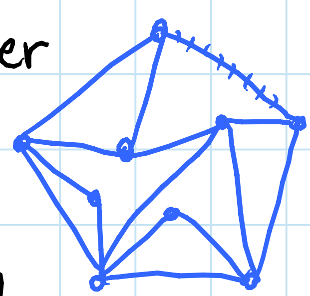
- strictly expansive (other than )
- one step in poly. time: convex program
- many steps; inaccurate (without projection)

  OPEN: how many? pseudopolynomial?

② pointed pseudotriangulations [Streinu 2000, 2005]

↳ maximal edge set on given points with >180° angle at every vertex

- expansive
- $n^{O(1)}$ steps
- one step follows 1 D.O.F. linkage → delete edge of convex hull
  - best algorithm is exponential
  - OPEN: are pseudotriangulations easier than general 2D linkages? (e.g. they are noncrossing)
- PROJECT: implement this algorithm



Algorithms for unfolding 2D chains: (cont'd)

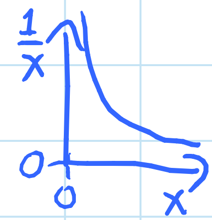
③ energy [Cantarella, Demaine, Iben, O'Brien 2004]

- not expansive
- one step is $O(n^2)$ & exact on real RAM
- pseudopolynomial number of steps
  ↳ poly. in n & r = max. dist./min. distance

Approach:

- define energy function on configurations:

$$E(C) = \sum_{\text{edge } vw} \; \sum_{\substack{\text{vertex } u \\ \neq v \text{ or } w}} \frac{1}{d(u, vw)}$$



- any energy-decreasing motion avoids crossings: approaching 0 dist. shoots $E \to \infty$
- expansive motion decreases energy (in fact, every term)

$\Rightarrow$ energy-decreasing motions exist

smooth $\Rightarrow$ downhill gradient of energy exists: $-\nabla E$
  - computable in $O(n^2)$ time

- lower bound gradient, upper bound curvature

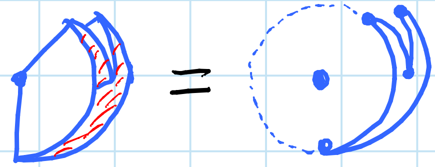
$\Rightarrow O(n^{123} r^{41})$ step bound (!)

OPEN: improve step bound (likely not hard)

OPEN: $n^{O(1)}$ step bound possible? conjecture no

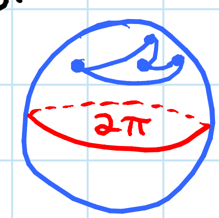
OPEN: is minimum-energy configuration unique? for equilateral polygons, it's a regular n-gon

Single-vertex rigid origami: [Streinu & Whiteley 2001]
every folded state of a single-vertex crease pattern can be folded rigidly
(continuously, faces staying rigid)



linkage folding!

Spherical Carpenter's Rule Theorem: [Streinu & Whiteley]
closed chain of total length 2π on unit sphere has a connected configuration space
- proof based on projective invariance of infinitesimal rigidity

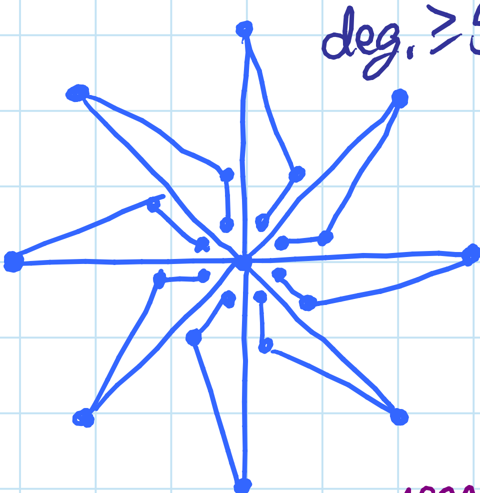


- length $\le 2\pi \Rightarrow$ lie in hemisphere
  $\Rightarrow$ can project to plane
- length $= 2\pi \Rightarrow$ convex config. = equator
- length $< 2\pi \Rightarrow$ 2 convex configs. (cw & ccw)
  $\Rightarrow$ 2 connected components of configs.
- length $> 2\pi \Rightarrow$ no convex configuration

Touching case (e.g. flat folding) handled by recent self-touching Carpenter's Rule Theorem
[Abbott, Demaine, Gassend 2007]

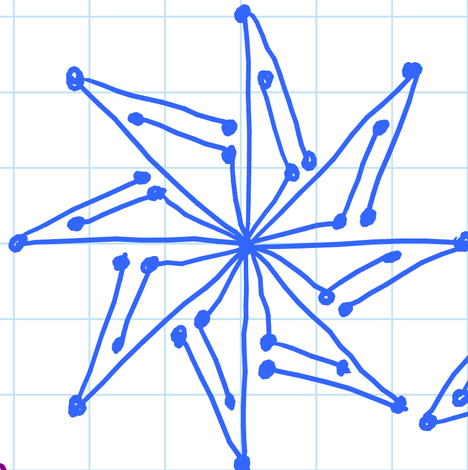
Locked 2D trees:

deg. ≥ 5



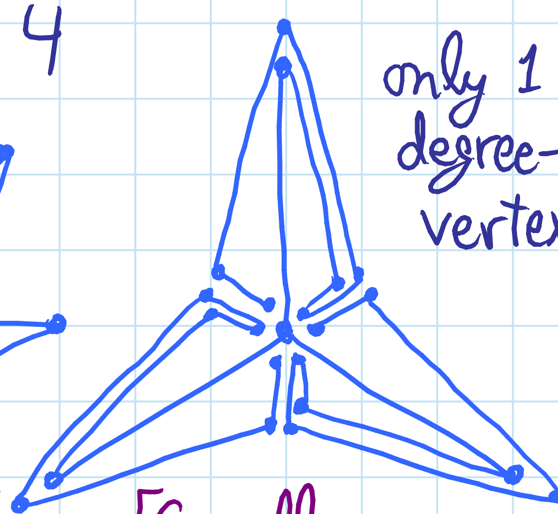
[Biedl et al. 1998/2002]

diameter 4

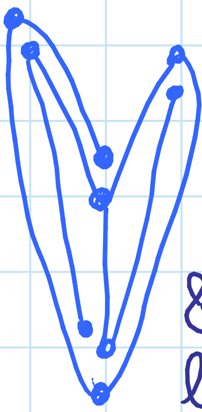


[Poon 2005]

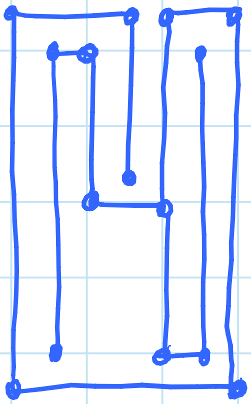
only 1 degree-3 vertex



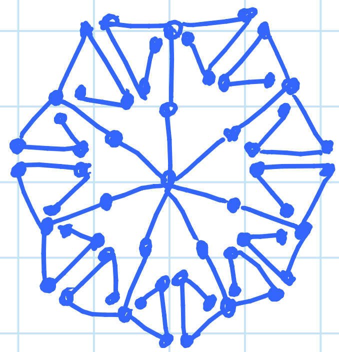
[Connelly, Demaine, Rote 2002]



8 edges linear



orthogonal



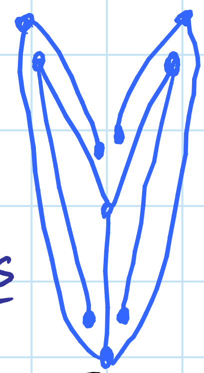
equilateral

not tight

[Ballinger, Charlton, Demaine, Demaine, Iacono, Liu, Poon 2009]

- linear = edges lie (nearly) in a line
- locked linear trees have
  - ≥ 8 edges
  - ≥ 9 edges or diameter ≥ 6

  [Ballinger et al.]



9 edges diam. 5

OPEN: 8 edges minimal for nonlinear?
14 edges minimal for orthogonal?

$\boxed{\text{OPEN}}$: characterize locked linkages
e.g. locked trees in 2D or chains in 3D
- polynomially solvable?
- special case: linear trees

Related problem: can you fold config. A → config. B?
- PSPACE-complete for 2D trees & 3D chains
[Alt, Knauer, Rote, Whitesides 2004]
- but their reductions use locked linkages as gadgets — so all locked

Infinitesimally locked linkages [Connelly, Demaine, Rote 2002]

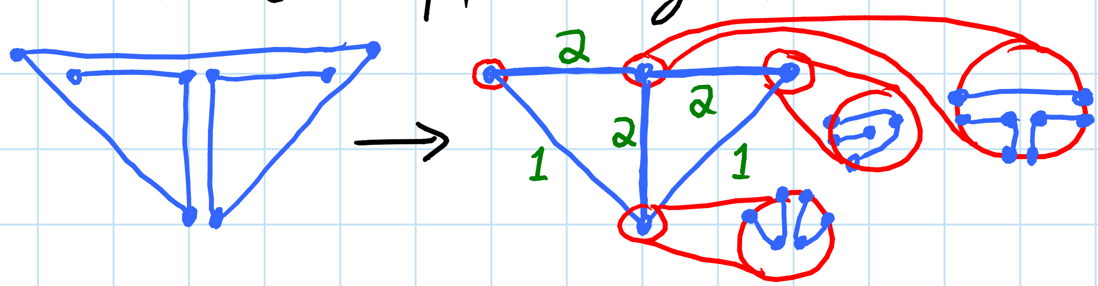
Intuition: in many locked examples (particularly 2D), as gaps get smaller, so do valid motions

Locked within ε = configuration from which it is impossible to get farther than ε in configuration space

Rigid = locked within $\emptyset$

— but trees are never rigid... right?

Self-touching configuration allows infinitesimal gaps: geometric overlap, distinguished combinatorially



— now can be rigid

Return to nontouching: rigidity $\Rightarrow$ "strongly locked"

Strongly locked = sufficiently small perturbations are locked within ε, for any $\varepsilon > 0$

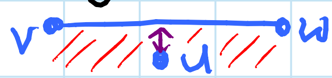
δ-perturbation = move vertices within δ-disks, preserving combinatorial sidedness

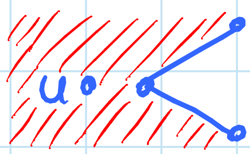
Every self-touching has a (non-self-touching) δ-perturbation, for all $\delta > 0$ [Ribó Mor, PhD 2006]

Proof based on "sloppy rigidity": [Connelly 1982]
if relax the edges in a rigid tensegrity
(bars can change length by δ
struts can shrink by δ, etc.)
then still can't move more than ε

Infinitesimally locked linkages: (cont'd)

Infinitesimal rigidity:

- implies rigidity
- "zero-length strut" (linear inequality):
  u should remain right of vw 

- sometimes nonconvex: 

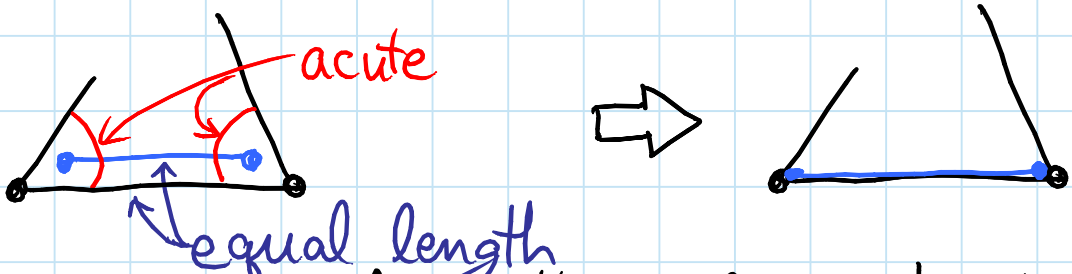
$\Rightarrow$ conservative polynomial test (drop constraints)
or exponential test (split into 2 convex)

- analogs of equilibrium stress & duality
- even Maxwell–Cremona [Ribó Mor, PhD 2006]
- nice proofs by hand: positive stress on struts
  + underlying linkage rigid
  ($\Rightarrow$ inf. rigid
  $\Rightarrow$ rigid
  $\Rightarrow$ strongly locked)

PROJECT: implement locked linkage tester/designer tool

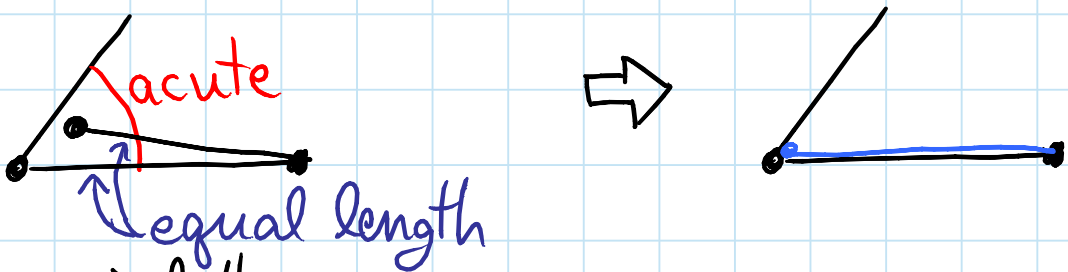
Infinitesimal locking rules: [Connelly, Demaine, Demaine, Fekete, Langerman, Mitchell, Ribó, Rote 2006]

Rule 1:



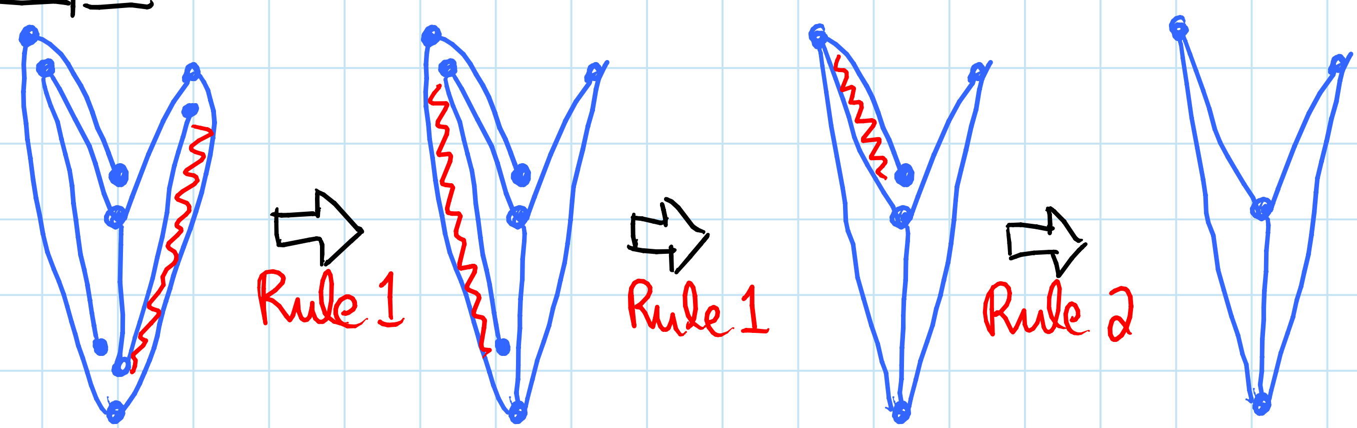
⇒ pinned together for positive time

Rule 2:



⇒ ditto

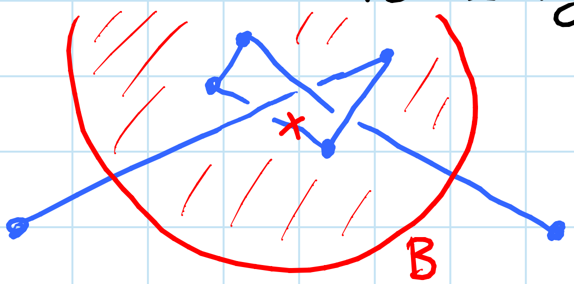
Example:



rigid

rigid
⟺ strongly locked

<u>3D knitting needles</u>: locked if each end bar is longer than Σ_i middle bars



[Cantarella & Johnston 1998]

<u>Proof</u>: draw ball B centered at midpoint of middle bars, diameter $= \Sigma_i$ middle bars $+ \varepsilon$

$\Rightarrow$ middle vertices remain inside B, end vertices remain outside B.

$\Rightarrow$ any motion could be augmented by an unknotted rope connecting two ends outside B.

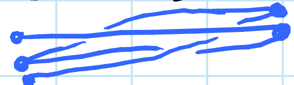
$\Rightarrow$ straightening motion would untie trefoil knot. $\square$

OPEN: minimum possible edge length ratio for which locked 3D chain exists?

– best example is $1 : 3 + \varepsilon$ above

OPEN: any locked <u>equilateral</u> 3D chain? [Biedl et al.]

equilateral 3D chain self-weaving on line [E. Demaine]



equilateral unknotted closed chains? [M. Demaine]

equilateral trees? [E. Demaine; Poon]

equilateral chain of equal-width cylinders? [O'Rourke]