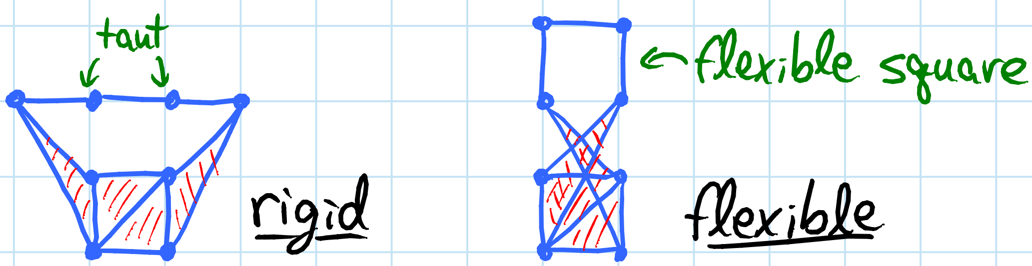
6.849	Lecture 9	Oct. 6, 2010

Review: (but without pinning)

Linkage = graph + lengths of edges $\ell : E \to \mathbb{R}^{>0}$

Configuration of linkage = coordinates for vertices $C: V \to \mathbb{R}^d$ satisfying edge lengths

Rigidity: a linkage configuration is
- flexible if there's a nontrivial motion starting there
  - → positive duration
  - → not a rigid motion (translation + rotation)
- rigid otherwise (Ø D.O.F. / isolated point)
- depends on configuration, not just linkage:



- testing for given configuration is coNP-hard [Abbott, Barton, Demaine 2008]

Applications:
- architecture [e.g. Why Buildings Fall Down]
- biology (which parts of atom-bond structure move?) → [Michael Thorpe]
- linkage folding [see L10]

Generic rigidity: rigidity is usually a property of the graph, not the configuration or linkage

Realization of a graph
= coordinates for vertices $C: V \to \mathbb{R}^d$ ($=$ pt. in $\mathbb{R}^{dn}$)
= configuration of a linkage of the graph

Generic realization avoids any "degeneracies":

e.g.
- 3 points collinear
- 4 points concyclic, or inducing parallel segs.

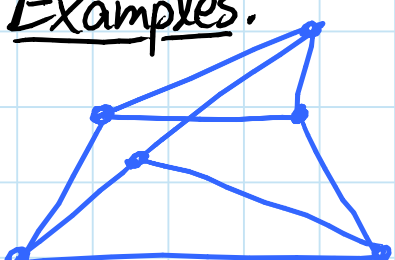
another definition in L10
- any nontrivial rational algebraic equation
  ↳ something not universally true — depends on input

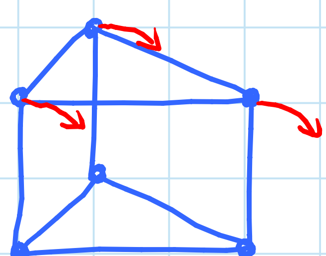
→ lower-dimensional subset of $\mathbb{R}^{dn}$
⇒ almost every realization is generic
i.e. random $\underbrace{\text{realization}}_{\text{or } \varepsilon\text{-perterbation}}$ is generic with probability 1

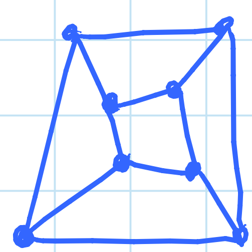
Rigidity is generic: for any graph, either
- all generic realizations are rigid
  ⇒ "<u>generically rigid</u>"

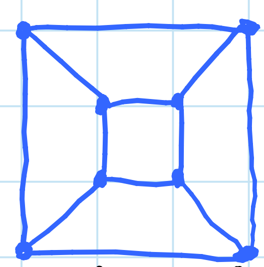
or
- all generic realizations are flexible
  ⇒ "<u>generically flexible</u>"

Examples:


gen. rigid


rarely flex.

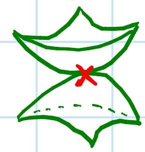

gen. flex.


rarely rigid

Minimally generically rigid graph = gen. rigid
+ removing any edge ⇒ gen. flexible
– characterize these first
– then gen. rigid graphs = supergraphs of these

Degree-of-freedom analysis: min. gen. rigid graph in 2D has exactly $2n-3$ edges (n vertices) & in dD has exactly $dn - \frac{d(d+1)}{2}$ edges

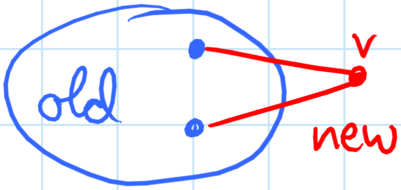
Proof: Empty graph has dn degrees of freedom.
Rigid graph has $\frac{d(d+1)}{2}$ degrees of freedom for rigid motion (2 transl. + 1 rotation in 2D)
Each edge imposes one $\underbrace{\text{algebraic constraint}}_{=(dn-1)\text{-dim. algebraic surface}}$, either:
– algebraically dependent: (redundant edge)
always holds given previous constraints
⇒ configuration space unchanged
or – algebraically independent:
almost never holds (only if degenerate)
⇒ decreases dimension by 1
(in degenerate cases, can reduce dimension by 0 or more than 1: surfaces meeting degenerately in fewer dimensions)

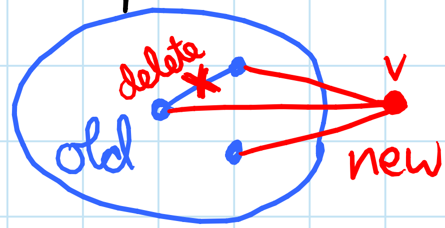


□

Henneberg characterization: [Henneberg 1911]

a graph is minimally generically rigid in 2D $\iff$ it can be built from •—• using two operations:

Type I: 

Type II: 

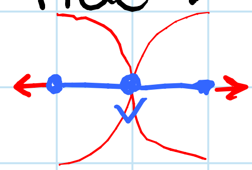
Proof: ($\Leftarrow$) holds in any dimension ~ degrees d & $d+1$

① G gen. rigid $\iff$ G′ resulting from Type I is

($\Leftarrow$) if G is gen. flexible

then v can "come for the ride":

(intersection of 2 circles)



generically v won't be taut:

($\Rightarrow$) if G is gen. rigid

then so is G′: can't just move v

② G min. gen. rigid $\iff$ G′ from Type I is

— similar: prove G−e flexible for all e $\iff$ G′−e flexible for all e

enough for ($\Leftarrow$) proof

③ G min. gen. rigid $\Rightarrow$ G′ from Type II is

— harder: need infinitesimal rigidity [L4]

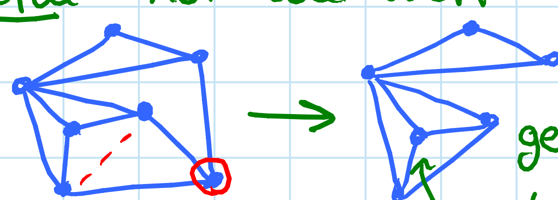
"converse" of ③

④ G′ min. gen. rigid with degree-3 vertex v

$\Rightarrow$ G′ is result of Type II of some min. gen. rigid G

— ditto

Careful: not all work! [Abbott]

min. gen. rigid  gen. flexible

overbraced

Proof of Henneberg characterization: (cont'd)

($\Rightarrow$) holds only in 2D

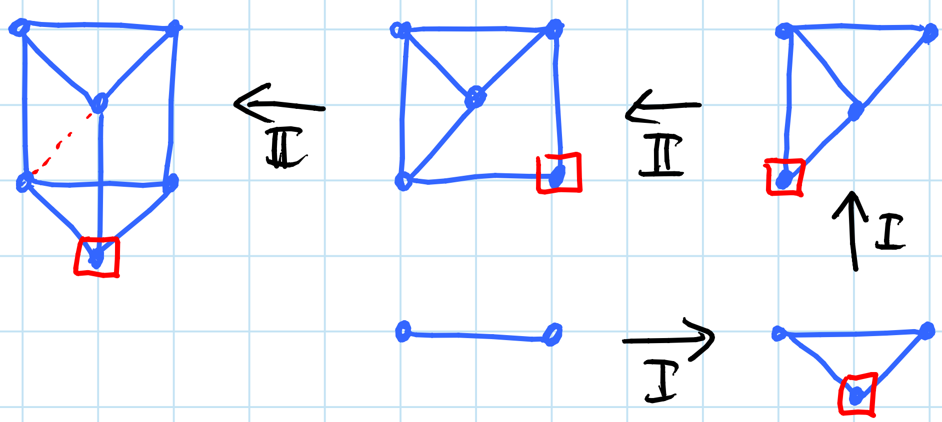
- DOF analysis $\Rightarrow$ $2n-3$ edges
- Handshaking Lemma: in any graph,
  Σ vertex degrees $= 2$ (# edges)

$\Rightarrow$ average degree $= 4 - \frac{6}{n} < 4$

$\Rightarrow$ some vertex v of degree < 4

- can't be degree 1 or else flexible
- if degree 2: delete v (anti-Type I op.)
  2 $\Rightarrow$ still min. gen. rigid
  induct
- if degree 3: 4 $\Rightarrow$ anti-type II op.
  induct

Example:



Laman's characterization: [Laman 1970]

a graph is minimally generically rigid in 2D $\iff$
it has $2n-3$ edges ($n>1$ vertices)
& every subset of k vertices induces $\le 2k-3$ edges

Proof: [based on Jack Graver's *Counting on Frameworks*]

($\Rightarrow$) Consider Henneberg construction.
Each operation adds one vertex ($n \nearrow 1$),
increases # edges by 2, and
any subset of vertices either
- excludes $v \Rightarrow$ # edges $\searrow$ 0 or 1

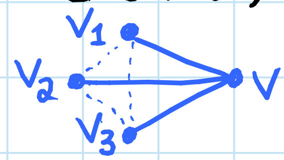
or - includes $v \Rightarrow k \nearrow 1$, #edges $\nearrow \le 2$

Base case: 

($\Leftarrow$) $2n-3$ edges, so again
there is a vertex v of degree <4
- can't be degree 1: then remaining $n-1$ vertices induce $2n-4$ edges
- if degree 2:
  - delete v (anti-Type I op.)
  - every subset of remaining vertices induces same # edges as before

  $\Rightarrow$ still Laman
  - induct & use 2

Proof of Laman characterization: (cont'd)



($\Leftarrow$) – if degree 3:

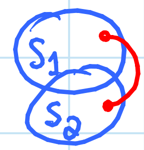
– some edge is missing among v's neighbors: 4 vertices induce $\leq 2\cdot 4-3=5$ out of 6 edges

– need to find anti-Type II op. preserving Laman

– any k vertices including v_1, v_2, v_3 & not v must have $< 2k-3$ edges ($\Rightarrow$ room for 1 more): adding v increases $k \nearrow 1$, #edges $\nearrow 3$

– suppose there are k_i vertices S_i $(i=1,2,3)$ containing v_{i+1}, v_{i+2} but not v & inducing $2k_i-3$ edges

– Inclusion–Exclusion principle:



$$E(S_1 \cup S_2) \geq E(S_1) + E(S_2) - E(S_1 \cap S_2)$$
$$= 2(k_1+k_2) - 6 - E(\underbrace{S_1 \cap S_2}_{\text{suppose} > 1 \text{ vertex}})$$
$$\geq 2(k_1+k_2-|S_1 \cap S_2|) - 3$$
$$= 2|S_1 \cup S_2| - 3$$

\# edges in induced subgraph

but $S_1 \cup S_2$ contains v_1, v_2, v_3 & not v $\Rightarrow$ impossible

$\Rightarrow S_1 \cap S_2 = \{v_3\}$, $S_2 \cap S_3 = \{v_1\}$, $S_1 \cap S_3 = \{v_2\}$

$\Rightarrow |S_1 \cup S_2 \cup S_3| = k_1 + k_2 + k_3 - 3$

& $E(S_1 \cup S_2 \cup S_3) \geq E(S_1) + E(S_2) + E(S_3)$
$$= 2(k_1+k_2+k_3) - 9$$
$$= 2|S_1 + S_2 + S_3| - 3$$

but $S_1 \cup S_2 \cup S_3$ contains v_1, v_2, v_3 & not v

$\Rightarrow$ can add some edge among v_1, v_2, v_3 & delete v & preserve Laman; induct. $\square$

$O(nm)$ algorithm: (n vertices, m edges)

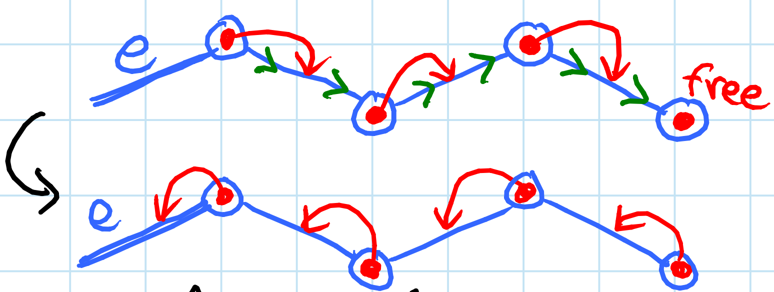
[Jacobs & Hendrickson 1997]

Edge set is <u>independent</u> if,
for each edge e & for each subset of k vertices,
induced edges + 3 copies of e $\le 2k$.

<u>Equivalent formulation of Laman:</u> a graph is generically rigid $\Leftrightarrow$ has $2n-3$ independent edges (forming minimally generically rigid subgraph)

<u>Pebble algorithm:</u>

- repeatedly try to add edge to independent set (outer loop of m iterations)
- give each vertex 2 pebbles, assignable to any 2 incident edges
- to see whether edge e can be added to indep. set:
  - add 4 copies of e ~ unpebbled
  - try to get a pebble on every edge by searching for directed path from e to free pebble:



<u>directed graph</u> connectivity — $O(n)$ time

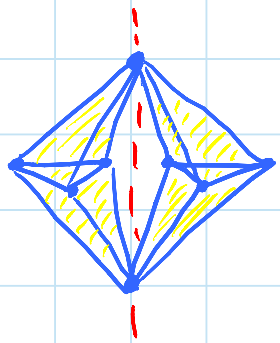
  - succeed precisely if e is independent
  - otherwise e is <u>redundant</u>: discard

OPEN: fastest 2D generic rigidity test?

- best so far: $O(n^2)$ [Hendrickson 1992]
- similar to above, using bipartite matching
- $O(n^{1.5}\sqrt{\log n})$ via decomposition into two disjoint spanning trees [Daescu & Kurdia 2009]
- unlikely to do better than matching... ($O(n^{1.5})$)
- can we use dynamic connectivity DSs?

OPEN: combinatorial test for 3D generic rigidity

- Laman condition ($3n-6$) still neccessary but insufficient:



double bananas

PROJECT: implement Laman and/or Henneberg

- colorcode over/underbracing, min. rigid subgraph
- show Henneberg construction

PROJECT: Henneberg computer game/puzzle

- player starts from •—• & tries to build target min. gen. rigid graph by Henneberg ops.
- fairly easy backwards; hard forwards!

Rigidity of polyhedra: (e.g. geodesic domes)
(see Connelly's 1993 survey)

Cauchy's Theorem: [Cauchy 1813; Steinitz 1934]
a polyhedron with rigid faces
has only one convex configuration
(modulo rigid motion)
- true even if faces are pieces of paper
(can add/remove creases) [Alexandrov 1958]

Dehn's Theorem: [Dehn 1916]
the edge skeleton of a (strictly) convex
polyhedron with triangular faces
is (infinitesimally) rigid

Alexandrov's Theorem: [Alexandrov 1958]
the edge skeleton of a triangulated
convex polyhedron is (infinitesimally) rigid

Bellows Theorem: [Connelly, Sabitov, Walz 1997]
when (nonconvex) polyhedron flexes,
its volume is preserved