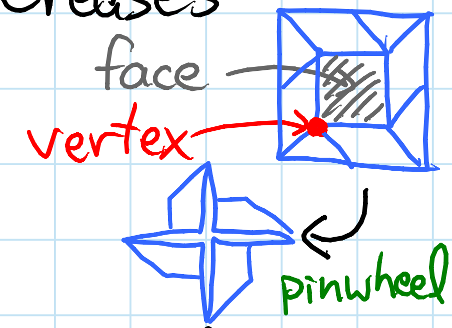
6.849	Lecture 2	Sept. 13, 2010

Origami terminology:

Piece of paper = 2D polygon (most often)
with distinguished top/bottom sides

Crease = line segment or curve on paper

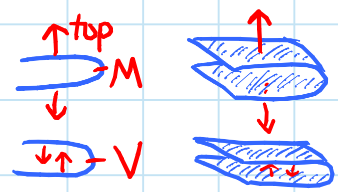
Crease pattern = collection of creases
= planar graph
drawn on paper



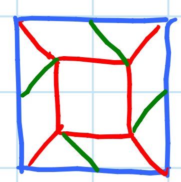
Folded state = finished origami
– unfolding → crease pattern

Flat folding = folded state lying in a plane
– call its crease pattern flat foldable
(must use all creases)

Mountain crease = bottom sides touch
Valley crease = top sides touch



Mountain-valley assignment = which creases
in crease pattern are mountain/valley



origami notation: —·—·—· / - - - - -

Mountain-valley pattern = crease pattern
+ mountain-valley assignment

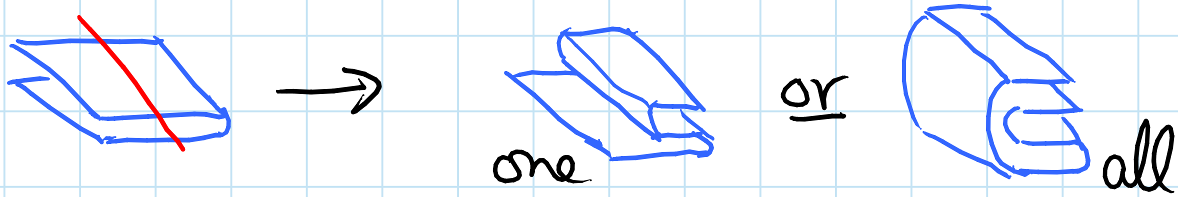
Example: crumple up a piece of paper

Simple fold: fold along a single line, by $\pm 180°$ (mountain/valley)

– choice of how many layers to fold

One-layer simple fold = just top or bottom

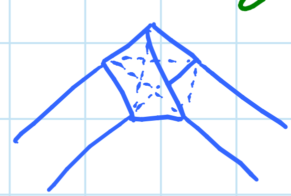
All-layers simple fold = all the way through



$\boxed{\text{TODAY is mostly about simple folds}}$

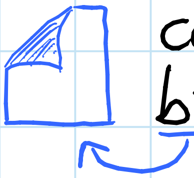
Strip: long narrow rectangle 

Example of nonsimple strip folding: tie a knot



Folding any shape: [Demaine, Demaine, Mitchell 2000]

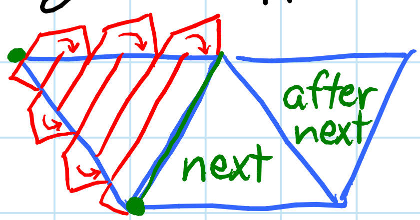
(a.k.a. silhouette [Bern & Hayes 1998] / gift wrapping [Akiyama/Gardner])



Every connected union of polygons in 3D, each with a specified visible color (on each side), can be folded from a sufficiently large piece of bicolor paper of any shape (e.g., square).

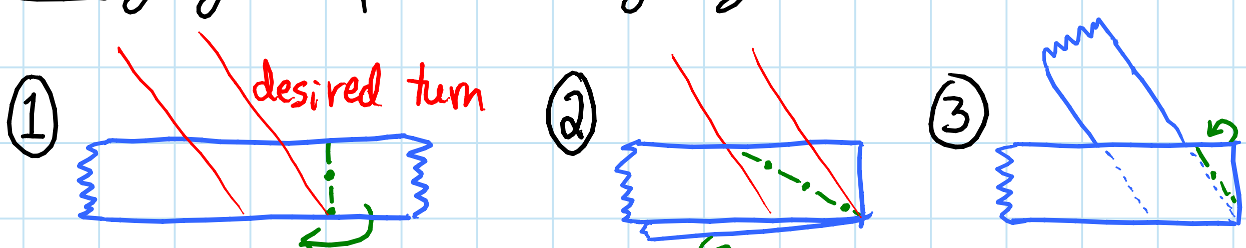
Proof: fold paper down to long narrow strip (!)

- triangulate the polygons
- choose a path visiting each triangle at least once
- cover each triangle along the path by zig-zag parallel to next edge, starting at opposite corner:



choose parity of zig-zag to arrive at correct corner for next triangle

- turn gadget implements zig-zags & vertex turns:



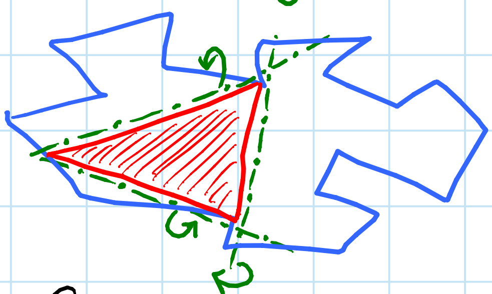
perpendicular mountain

fold bottom layer

hide excess (many folds)

Proof of folding any shape: (cont'd)

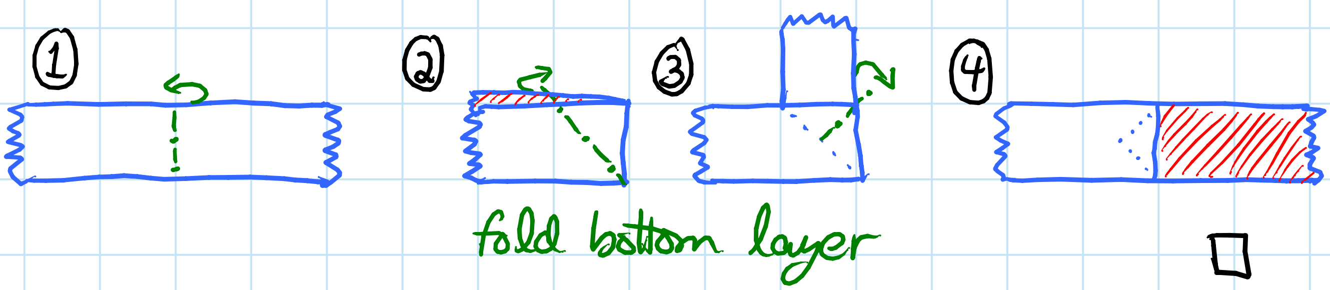
- hide excess paper underneath each triangle:
  (more generally, can hide under any convex polygon)



repeatedly mountain fold along lines extending desired edges

  - if paper is unicolor (or don't care) can use valley folds $\Rightarrow$ simple folds
  - if mountain folds, might colide with other Δs $\Rightarrow$ not really simple folds (but still works as origami fold)

- color-reversal gadget along transition between triangles of opposite colors:

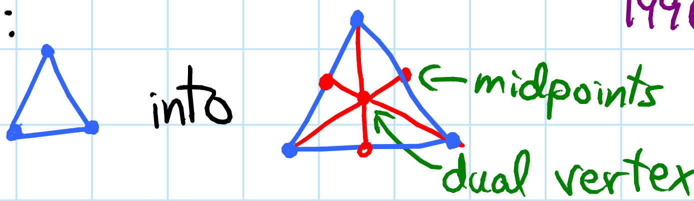


□

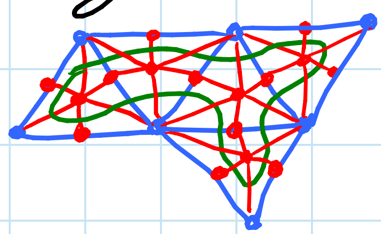
Pseudo-efficiency: if allowed to start with any rectangle of paper, then can achieve area(paper) = area(surface) + ε for any $\varepsilon > 0$

Proof: construct **Hamiltonian refinement** [Arkin et al. 1996] of triangulation:

- cut each $\triangle$ into



- walk around spanning tree of original dual:



- now visit each triangle exactly once
- wastage from turns $\to 0$ with strip width. $\square$

Alternate proof from class: (should work)

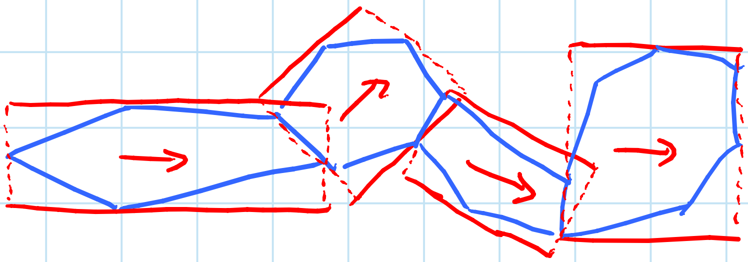
- visit $\triangle$s in any order, but when traversing from one to next, go directly (without covering (much of) intervening $\triangle$s
- wastage is $\approx$ strip width $\cdot \sum$ diameter($\triangle$) $\to 0$ as strip width $\to 0$

$\boxed{\text{OPEN}}$: pseudopolynomial upper bound? lower bound?

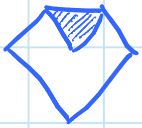
(uncovered :)

Seam placement: can place seams (visible creases/ paper boundary) as desired, provided regions between seams are convex

- idea: vary strip width, use hide gadget



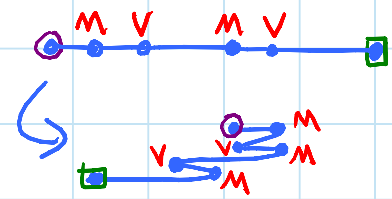
$\boxed{\text{OPEN}}$: what seam placements are possible?



1D flat folding: [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2004]

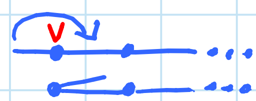
Piece of paper = line segment
Crease = point on paper
Flat folding lies on a line

All crease patterns are flat foldable:
zig-zag / accordion fold

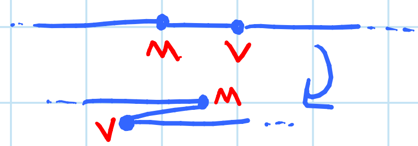


Not all mountain-valley patterns:

Two folding operations: (both simple)
① end fold if end length ≤ neighbor

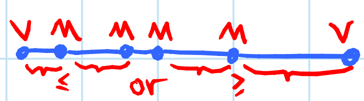


② crimp two consecutive creases
if length between ≤ both neighbors
& one mountain, one valley



Characterization:
mountain-valley pattern is flat foldable
⟺ there's a sequence of crimps & end folds

Tool: Mingling: for any maximal sequence of M's or V's,
adjacent V or M or end on at least one side
is nearer than adjacent M or V:

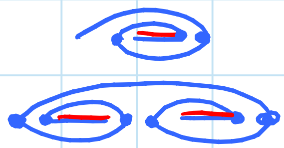


$\boxed{\text{OPEN}}$: proof without mingling?

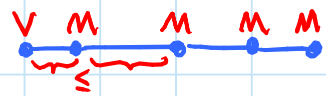
1D flat folding (cont'd)

Proof of characterization:

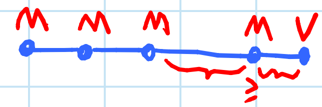
- flat foldable ⇒ mingling:
  - sequence of M's or V's form a spiral (or double spiral)



  - at least one end must be short
- mingling ⇒ end fold or crimp possible
  - for each maximal sequence of M's or V's
    write (if "left mingling"
    [otherwise



    & write) if "right mingling"
    ] otherwise



  ⇒ (] or [) or ()
  -)(⇒ crimp
  - leading (/ trailing) ⇒ end fold
  - if neither: [)[)···[)(](]···(]

    crash

- crimp/end fold preserves flat foldability
  - take flat folding
  - move some layers out of crimp

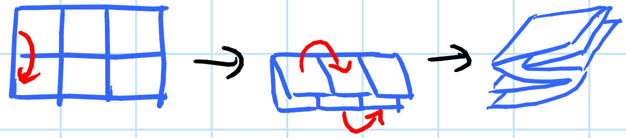
  ⇒ could start with crimp

- induct ⇒ sequence of crimps & end folds
  ⇒ flat foldable again. □

2D map folding: [Arkin et al 2004]

↳ rectangular paper with axis-parallel creases

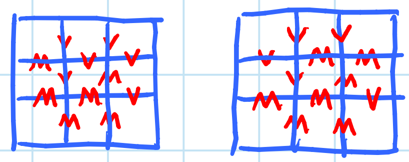
- again every crease pattern is flat foldable: zig-zag in x then y



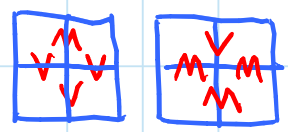
OPEN: characterize flat-foldable mountain-valley patterns — even $2 \times n$! [Edmonds 1997]

Simple folds are not as powerful in 2D:

(in contrast to 1D, where we can simulate crimp/end folds)



Characterization of simple foldability of maps:

- if simply foldable, must be a uniform horizontal/vertical line ↳ all M or all V
- crossing vert./horiz. lines must switch M⇔V here:
  - local 2×2 patterns:  & rotations

⇒ all uniform horizontal lines must be folded before any vertical lines become uniform, etc.

⇒ sequence of 1D problems on current uniform lines

- linear-time algorithm — $O(mn)$ for $m \times n$ map — by maintaining uniformity for each line, crimpability & end foldability on lengths