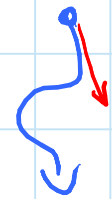
6.849	Lecture 10	Oct. 13, 2010

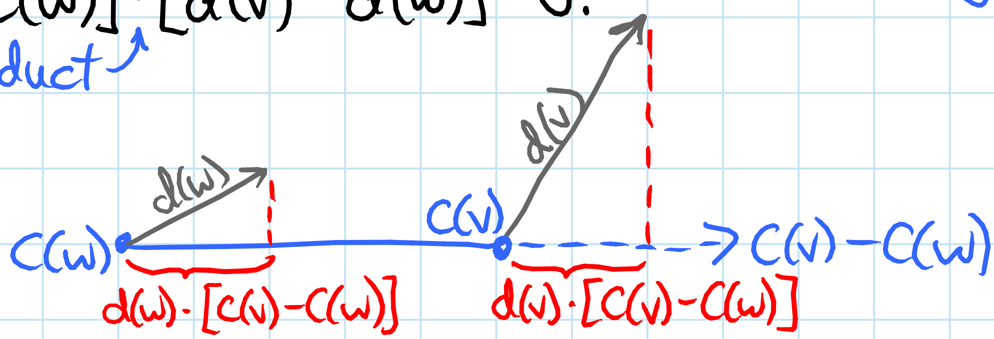
Infinitesimal rigidity: rigidity to the first order

Infinitesimal motion of a linkage configuration C
= valid first derivative of a motion w.r.t. time, at time $\emptyset$
= velocity vector $d(v)$ for each vertex v preserving edge lengths to the first order:

$$[C(v)-C(w)]\cdot[d(v)-d(w)]=0.$$



$a\cdot b \longrightarrow$ dot product
$= a_x\cdot b_x + a_y\cdot b_y$



Rigidity matrix: "everything is linear, to the first order"
edge-length constraints form a linear system:

$$\text{row per edge}\left\{\underbrace{\begin{pmatrix} 0\cdots 0 & C_x(v)-C_x(w) & C_y(v)-C_y(w) & 0\cdots 0 & C_x(w)-C_x(v) & C_y(w)-C_y(v) & 0\cdots 0 \end{pmatrix}}_{dn \text{ columns } (d \text{ dim., } n \text{ vertices})}\right. \cdot \begin{pmatrix} d_x(v_1)\\ d_y(v_1)\\ \vdots \\ d_x(v_n)\\ d_y(v_n)\end{pmatrix} = \emptyset$$

RIGIDITY MATRIX R

Infinitesimal motions = d for which $R\cdot d=\emptyset$
= kernel R = nullspace R
$\leftrightarrow$ linear subspace of some dimension: nullity R

Infinitesimally rigid if nullity $R = \underbrace{\binom{d+1}{2}}_{\mathbb{R}^d \text{ rigid motions}} - \underbrace{\binom{d-k}{2}}_{\text{symmetries}}$ where config. lies in k-dim. subspace

(correction by student Tony Zhang in Spring 2017) useful edges — degrees of freedom

<u>Rank-Nullity Theorem:</u> rank R + nullity R = # cols. = $d \cdot n$

$\Rightarrow$ inf. rigid $\Leftrightarrow$ rank $R = d \cdot n - \binom{d+1}{2} + \binom{d-k}{2}$ "full rank"

$\Rightarrow$ can test inf. rigidity in polynomial time using e.g. Gaussian elimination

<u>Generic point set</u> (more explicit definition than L9)

= all minors of rigidity matrix of complete graph

induced square submatrix on subset of rows & cols.

with nonzero determinant for some point set (i.e. not identically zero, algebraically)

are nonzero for <u>this</u> point set

<u>Generic results:</u>

- almost every configuration is generic
- at generic configurations, rigidity = infinitesimal rigidity = generic rigidity [L9]

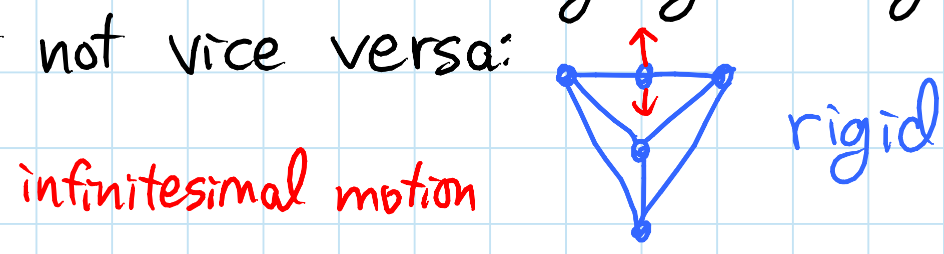
$\Rightarrow$ randomized polynomial-time generic rigidity test: test infinitesimal rigidity of random realization

- if any realization is infinitesimally rigid then graph is generically rigid (else generically flexible with probability 1)

<u>Taking derivatives:</u> flexible $\Rightarrow$ infinitesimally flexible

i.e. infinitesimally rigid $\Rightarrow$ rigid

- but not vice versa:

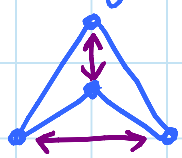


Tensegrity = tens(ional int)egrity [Snelson 1968; R. Buckminster Fuller]

$\boxed{\text{PROJECT}}$: build tensegrity sculpture

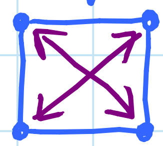
= linkage but where each edge is either:
- bar (as before): fixed length
- cable: length can only decrease (string/elastic/spider web)
- strut: length can only increase

- configuration space becomes $\underbrace{\text{semi-algebraic}}_{\text{polynomial inequalities}}$
- motion, rigidity as before
- but not generic rigidity:



flexible

vs.



rigid

- infinitesimal motion (& rigidity):

$$[C(v)-C(w)]\cdot[d(v)-d(w)] \begin{cases} =0 & \text{for bars } vw \\ \le 0 & \text{for cables } vw \\ \ge 0 & \text{for struts } vw \end{cases}$$

$\Rightarrow$ inf. motion space is a polyhedral cone

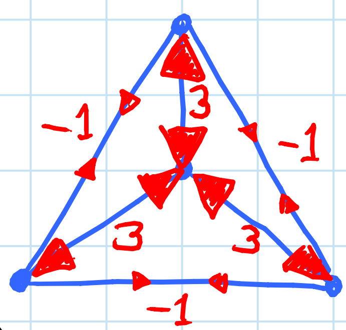
$\Rightarrow$ inf. rigidity testable in polynomial time via linear programming

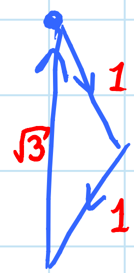
Equilibrium stress = real number for each edge $s: E \to \mathbb{R}$
such that $s(e) \geq 0$ for cables e (push back in resistence)
$s(e) \leq 0$ for struts e

EQUILIBRIUM → $\displaystyle\sum_{w:\text{edge } vw} s(vw) \cdot [C(v) - C(w)] = 0$
for all vertices v.

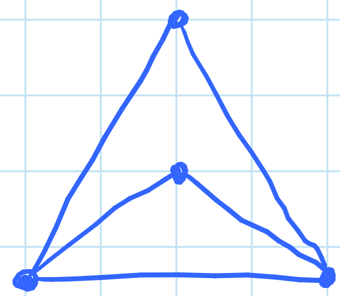
- view $s(vw)$ as a scale factor on force along edge vw felt equally by v & w.
  - $s(vw) > 0 \Rightarrow$ push on v & w (resist compression)
  - $s(vw) < 0 \Rightarrow$ pull on v & w (resist expansion)
  - $s(vw) = 0 \Rightarrow$ no force
- trivial equilibrium stress: $s(e) = 0$ for all e

Example:





vs.



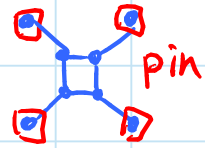
equilibrium stress | force polygon | no nontrivial equi. stress

Duality in tensegrities: [Roth & Whiteley 1981]

- some equilibrium stress is nonzero on strut/cable e
  ⇔ every infinitesimal motion holds e's length fixed
- tensegrity is infinitesimally rigid
  ⇔ every strut/cable is nonzero in some equilibrium stress
  & corresponding linkage is rigid
  ↳ replace cables & struts with bars

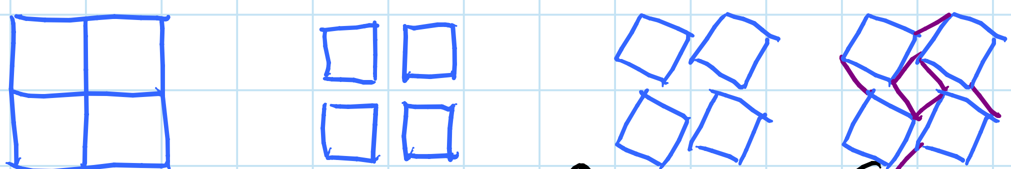
  } not necessary for "spiderweb" (next page)
- proofs based on linear-programming duality

Spiderwebs: all-positive equilibrium stress
(except on boundary)
⇒ infinitesimally rigid [Connelly 1982]



Origami tessellations:

- much history [Momotani 1984; Fujimoto 1982; Huffman 1960s, 1978; Resch 1968; Barreto 1997; Palmer 1997; Bateman 1990s; Verrill 1990s; Lang 2000s; Gjerde 2009]
- shrink-rotate algorithm:

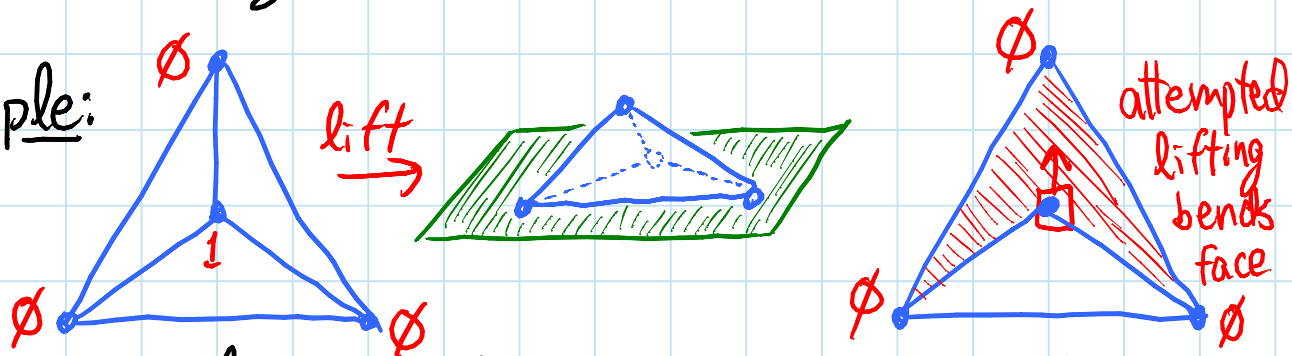


(0) tessellate (1) shrink (2) rotate (3) fill (4) fold

- works for some shrink/rotate amounts
  ⇔ tessellation is a spiderweb
  [Lang & Bateman 2010]

Polyhedral lifting of a **noncrossing** configuration
= z coordinate for each vertex $\quad z: V \to \mathbb{R}$
such that each face remains planar
– assume outside face at $z=0$ by rigid motion
– **trivial** lifting: $z(v)=\emptyset$ for all v

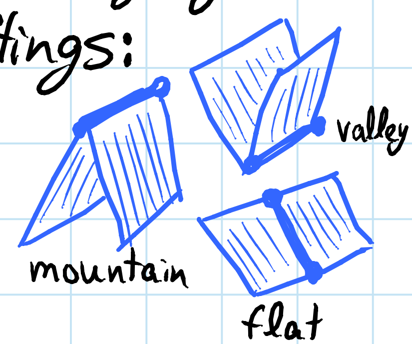
Example:



polyhedral lifting

no nontrivial poly. lifting

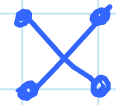
Maxwell–Cremona Theorem: [Maxwell 1864; Cremona 1872]
one-to-one correspondence in noncrossing tensegrity between equilibrium stresses & polyhedral liftings:
– negative stress $\leftrightarrow$ valley edge
– positive stress $\leftrightarrow$ mountain edge
– zero stress $\leftrightarrow$ flat edge



PROJECT: implement program to illustrate
stress $\leftrightarrow$ lifting correspondence
and/or stress $\leftrightarrow$ inf. motion correspondence

PROJECT: virtual tensegrity building toy
– illustrate infinitesimal flexibility if any

Noncrossing linkages:

- configuration cannot have crossing edges 
- config. space smaller; still semi-algebraic

Locked linkage if config. space is disconnected
i.e. no motion between some two configurations

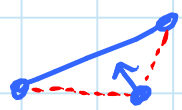
- summary:

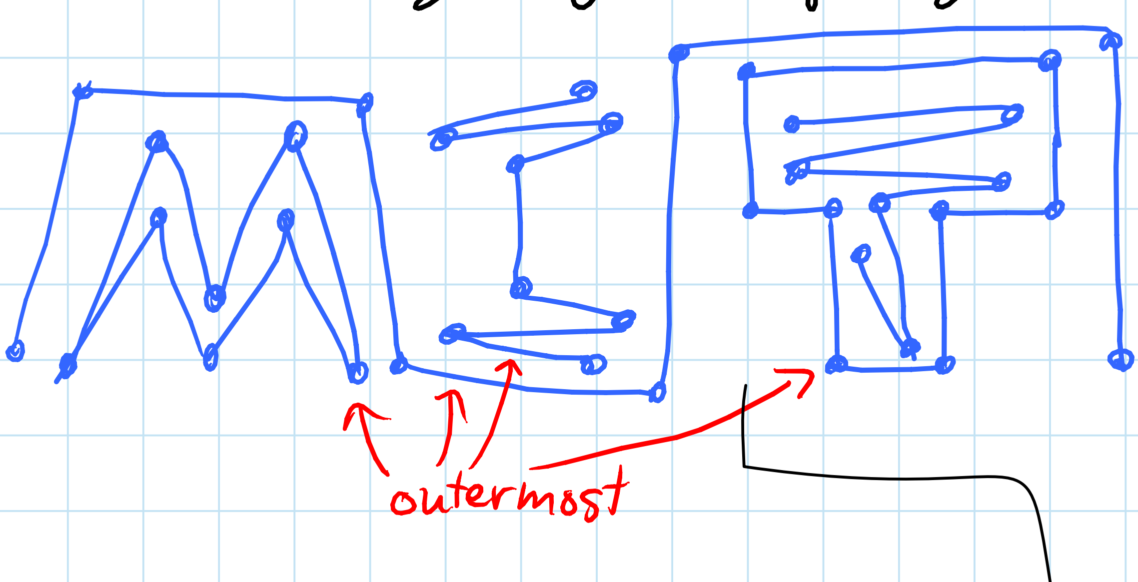
[L1]

	chains	trees
2D	never locked	can lock
3D	can lock	can lock
4D+	never locked	never locked

Carpenter's Rule Theorem: [Connelly, Demaine, Rote 2000/2003]
any linkage configuration of maximum degree 2 has a motion that

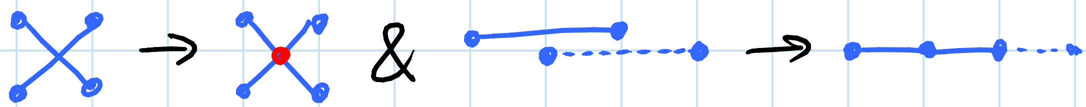
- straightens/convexifies all outermost open/closed chains
- is expansive: distance between any two vertices only increases

⇒ is noncrossing, by Δ inequality 

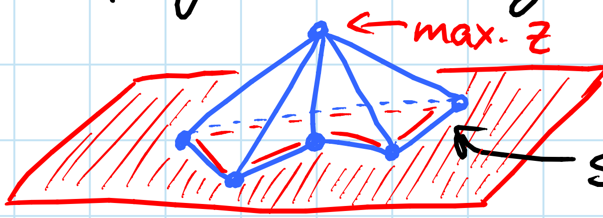


Proof sketch of Carpenter's Rule Theorem:

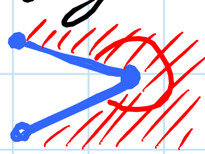
- build tensegrity from linkage (edges → bars)
  + all possible struts (except where bar exists)
- linkage has expansive infinitesimal motion
  ⇔ tensegrity is infinitesimally flexible
  ⇐ every equilibrium stress is zero (on struts)
  ⇔ every polyhedral lifting is flat (on struts)
  - detail: need to show stresses are equivalent in tensegrity vs. planarized tensegrity



  - here is where nested components get discarded
- slice hypothetical polyhedral lifting near max. z:
  - peak case:



    - convex vertices ↔ mountain edges
    - reflex vertices ↔ valley edges
    - every polygon has ≥3 convex vertices
    ⇒ ≥3 incident mountains (positive stress)
    ⇒ ≥3 incident bars (no cables)
    - but max. degree 2
  - general case:



← flat: need convex vertex to reach around reflex angle

    ⇒ flat except inside convex polygons
- integrate ordinary differential equation → expansive motion