

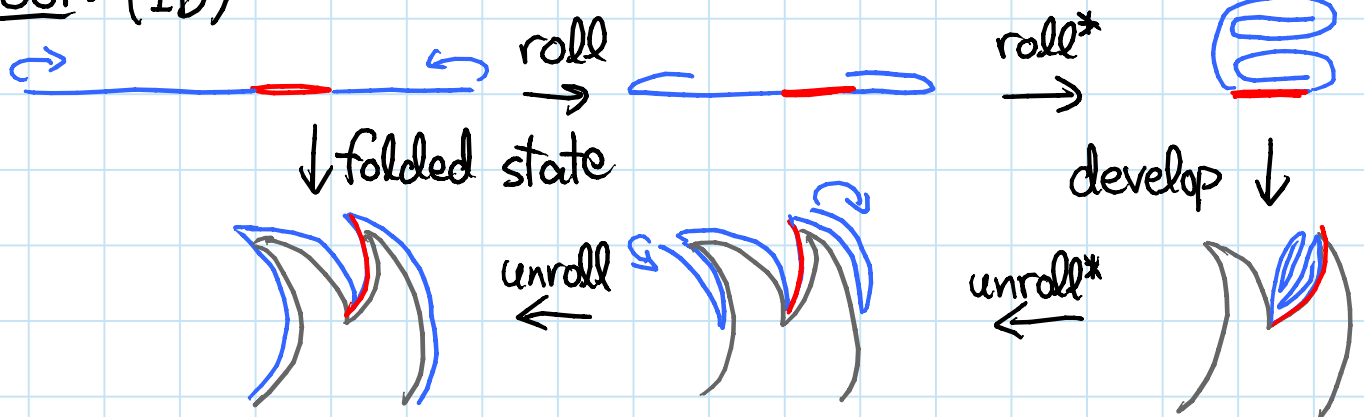
2 meanings of "folding": (origami)

- folded state = description of paper after folding
- folding motion = continuum of folded states
- we've focused on states, but in reality want motion

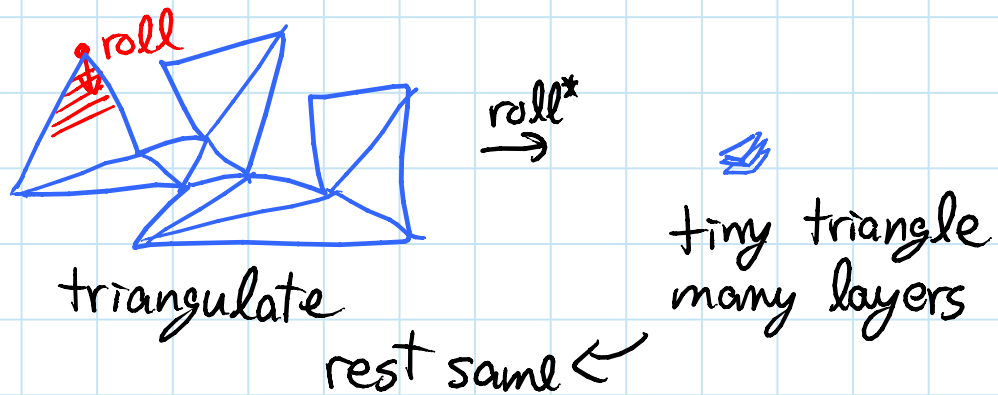
Equivalence: [Demaine, Devadoss, Mitchell, O'Rourke 2004]

any simple polygonal piece of paper has a folding motion into any desired folded state

Proof: (1D)



(2D)



□

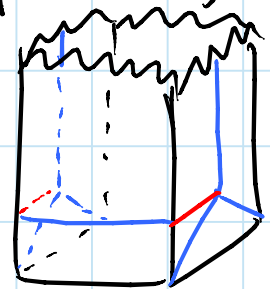
OPEN: what if paper has holes?
unknotted polyhedral paper? (for flattening)

OPEN: do finite number of extra creases suffice,
if target folded state does not touch itself?
- above, all points become crease points

Rigid origami: what folds without extra creases?
- faces of crease pattern = rigid polygons
- creases = hinges

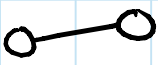
Example: [Balkcom, Demaine, Demaine, Ochsendorf, You 2006]
paper shopping bag doesn't fold rigidly
(for $h > w/2$, standard crease pattern)
height width

- 2 folded states: open & flat
- no folding motions



Little known about rigid origami:
looking for good open questions

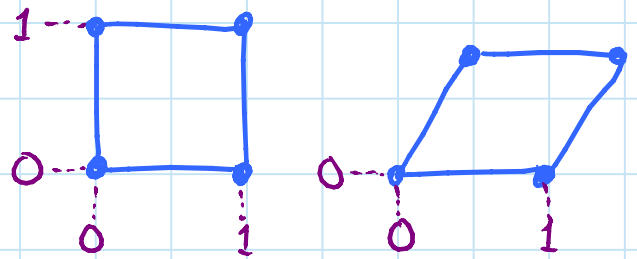
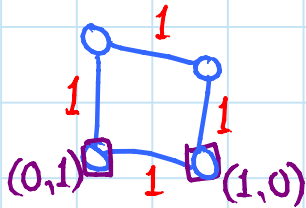
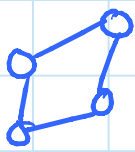
LINKAGES:

Graph = vertices (V) & edges (E) 
(connectivity / combinatorial structure)

Linkage = graph + lengths of edges ($l: E \rightarrow \mathbb{R}^{>0}$)
(intrinsic geometry)
[+ coordinates for pinned vertices ($p: V' \rightarrow \mathbb{R}^d$)]

Configuration of a linkage in \mathbb{R}^d
= coordinates for vertices ($C: V \rightarrow \mathbb{R}^d$)
satisfying constraints of linkage
($\|C(v) - C(w)\| = l(v,w)$ for all $\{v,w\} \in E$;
 $C(v) = p(v)$ for all $v \in V'$)
(allowing intersections for this lecture)

Example:



graph

linkage

two configurations

Motion (of a linkage in \mathbb{R}^d)
= continuum of configurations ($m: [0,1] \rightarrow \mathcal{C}$)

Configuration space = all configurations of a linkage

- view configuration of n -vertex linkage in \mathbb{R}^d as (special) point in \mathbb{R}^{dn} :

$$C = (\underbrace{\dots, \dots, \dots}_{d \text{ coords for } v_1}; \underbrace{\dots, \dots, \dots}_{v_2}; \dots; \underbrace{\dots, \dots, \dots}_{d \text{ coords. for } v_n})$$

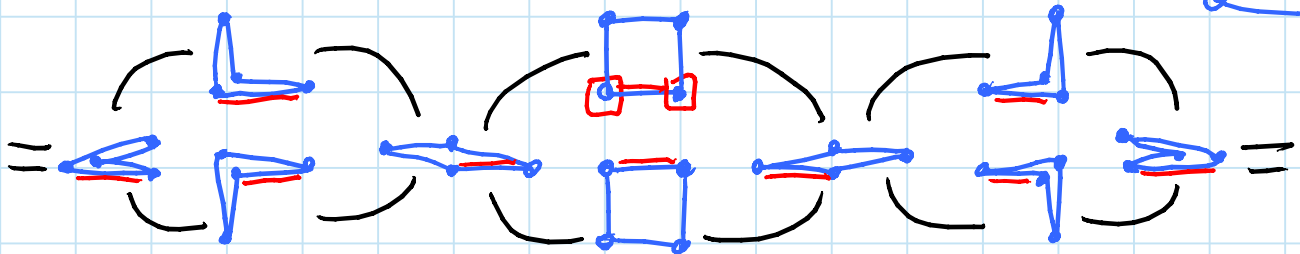
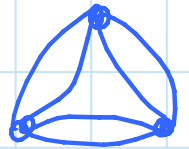
\Rightarrow configuration space = subspace of \mathbb{R}^{dn}

- motion = path/curve in configuration space
- square example: $n=4, d=2$

\Rightarrow configuration space lives in \mathbb{R}^8

- 4 dimensions fixed by pinning

- locally one dimensional; topologically:



Degrees of freedom = local intrinsic dimension of configuration space around configuration

- intuitively: $d \cdot (\# \text{ unpinned vertices}) - (\# \text{ edges})$
- (but in reality, some edges are extraneous - see L3)

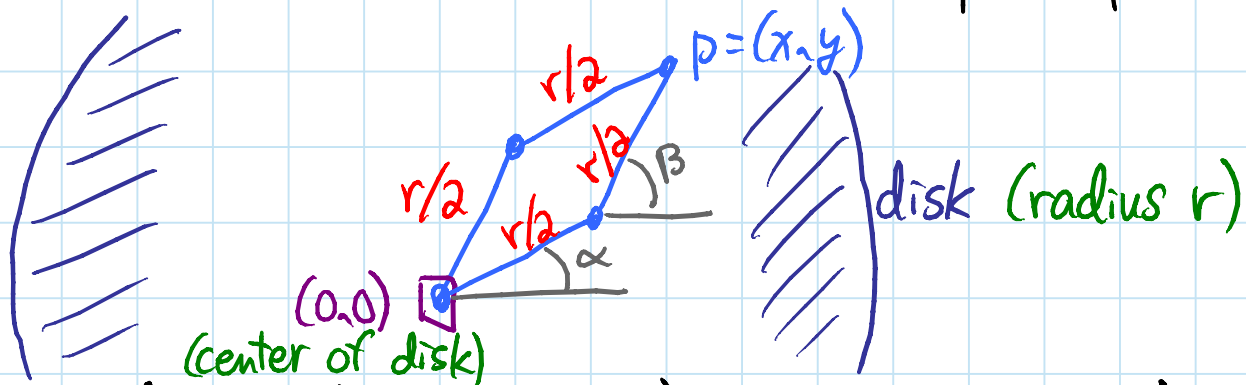
Trajectory of a vertex in a linkage = all points that vertex can reach in configurations (= projection of configuration space onto vertex's coords)

Kempe's Universality Theorem: [Kempe 1876 had bug;
Thurston; King 1999; Kapovich & Millson 2002;
Abbott, Barton, Demaine 2008]

Any algebraic planar curve $\varphi(x,y) = \sum_i c_i x^{p_i} y^{q_i} = 0$,
intersected with any bounded disk, (necessary)
is exactly the trajectory of a vertex of some linkage.

Kempe's "proof":

- start with rhombus to constrain point p within disk:



- goal: constrain $p=(x,y)$ to satisfy $\varphi(x,y)=0$

Main trick: use trig. to effectively "take logarithm"

$$- x = \frac{r}{2} \cos \alpha + \frac{r}{2} \cos \beta$$

$$- y = \frac{r}{2} \sin \alpha + \frac{r}{2} \sin \beta = \frac{r}{2} \cos(\alpha - \frac{\pi}{2}) + \frac{r}{2} \cos(\beta - \frac{\pi}{2})$$

- apply trig. identity

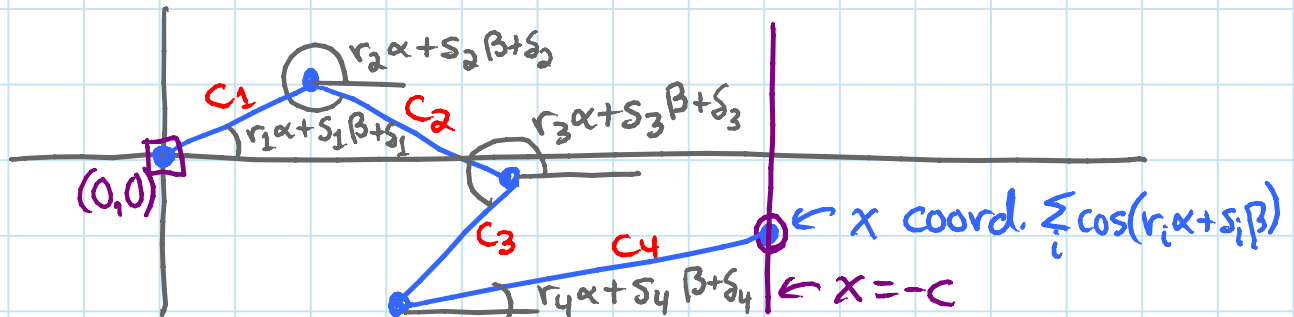
$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

to polynomial $\varphi(x,y) = \sum_i c_i x^{p_i} y^{q_i}$

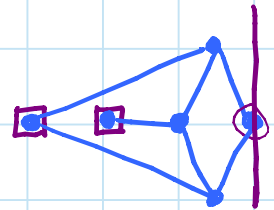
$$\Rightarrow \varphi(x,y) = \underbrace{c}_{\text{const.}} + \sum_i \underbrace{c_i}_{\text{const.}} \underbrace{\cos}_{\text{int.}}(\underbrace{r_i \alpha + s_i \beta}_{\text{int.}} + \underbrace{\delta_i}_{0 \text{ or } \pm \pi/2})$$

Kempe's "proof": (cont'd)

- new goal: construct line segment of length c_i & angle $r_i\alpha + s_i\beta + \delta_i$ for each i



- force final vertex on line $x = -c$ via large Peaucellier linkage

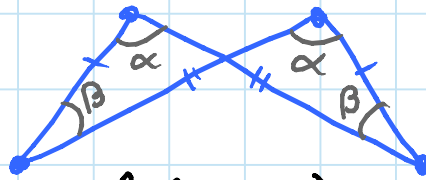


- build "machine" for angle arithmetic with ops.:
 - multiply given angle by integer
 - add two given angles
 - copy an angle from one place to another

Kempe's gadgets:

Contraparallelogram:

- opposite sides equal & self-crossing (not parallelogram)
- \Rightarrow opposite angles equal; α determines β



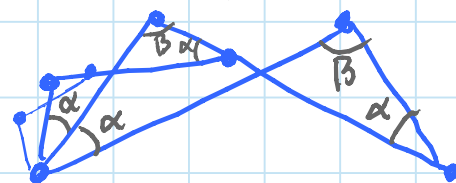
Multiplicator:

- k similar contraparallelograms sharing their β 's \Rightarrow equal α 's
- can be more efficient —

$O(\lg k)$ edges — by repeated doubling, but this will not affect final complexity

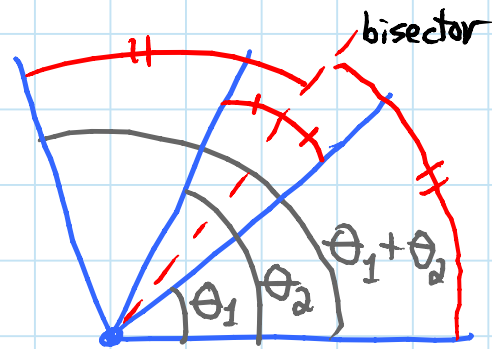


$k=2$



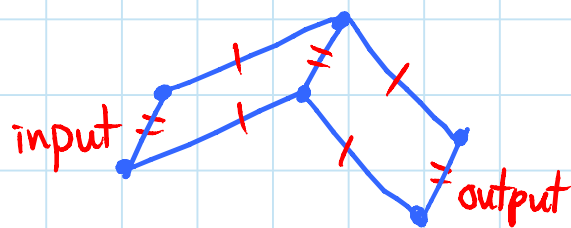
Additor:

- use 2x multiplicators to
 - bisect angle between segments
 - reflect x axis through bisector



Translator: two parallelograms

- opposite edges parallel & same length
- make adjacent edges long (& same) for reach
- could use big rhombus — but this construction allows arbitrary length of input (or output) edge



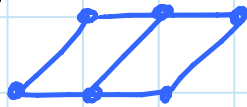
Bug: [Kapovich & Millson]

- parallelograms can flip to contraparallelograms & vice versa via degenerate (flat) configuration

⇒ Kempe proved weaker result:

trajectory includes desired poly. curve & more

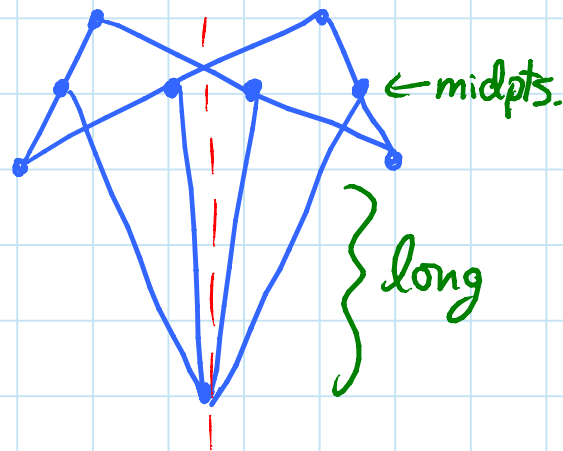
- fix for parallelogram:



- different, messier construction for complex polynomials

- fix for contraparallelogram:

[Abbott & Barton 2004]



Application:

Sign name via Weierstrass approximation theorem:

any continuous function $f: [a, b] \rightarrow \mathbb{R}$ has an ϵ -approximate polynomial p — $|p(x) - f(x)| \leq \epsilon$ for all $x \in [a, b]$ — for any $\epsilon > 0$

(apply to each coordinate of curve)

Generalizations/strengthenings:

- curves/surfaces in d dimensions
 - $\Theta(n^d)$ bars is optimal for degree n
 - any compact semialgebraic set (d -dim.)
(bounded system of polynomial \leq inequalities)
as vertex trajectory
 - coNP-hard to test rigidity
 - configuration space = union of finitely many
analytically isomorphic copies of any
desired algebraic set (any # dim.)
- mapping & inverse have local power-series expansion

[Abbott, Barton,
Demaine 2008]

[Kapovich
& Millson]

OPEN: what if edges are forbidden from crossing?
[Shimamoto 2004]

PROJECT: implement Kempe applet

PROJECT: sculpture based on Kempe linkage/gadgets

PROJECT: design linkages for letters of alphabet
(e.g. letter C: <http://www.iimloy.com/cindy/cindy.htm>)

Application: constructing algebraic numbers
in origami via alignments

[GFALOP 19.5: cf. Alperin & Lang 2006]