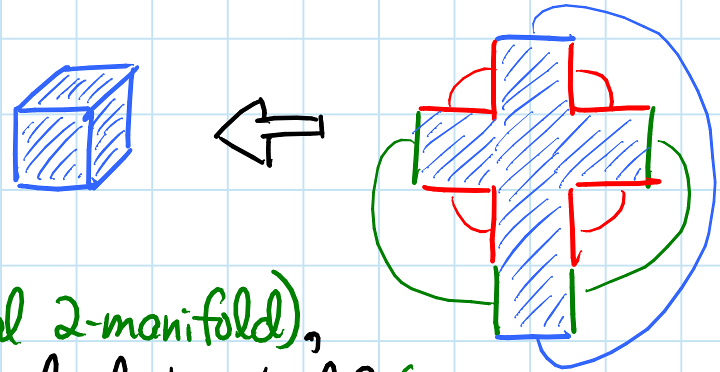
6.849	Lecture 15	Nov. 1, 2010



Folding polyhedra:

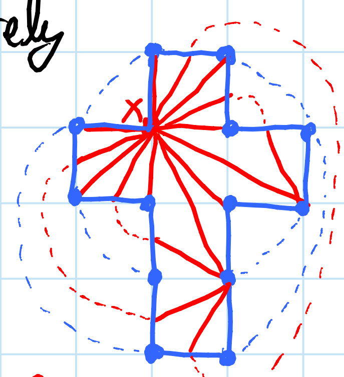
Decision problem:
given a polygon
(or connected metric polygonal 2-manifold),
can its boundary be glued to itself (in pairs of intervals)
such that resulting surface can be folded
into exactly a convex polyhedron?
↳ no multiple layers like origami

Enumeration problem: list all gluings & foldings

Combinatorial problem: how many can there be?

Why convex polyhedra? always possible to
fold into a (nonconvex) polyhedron
provided orientable or some unglued boundary
[Burago & Zalgaller 1960, 1996; O'Rourke 2010]

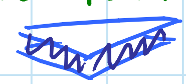
Alexandrov gluing: polygon + gluing induce a metric
by shortest-path lengths between all pairs of points
- metric is polyhedral: all but finitely many points have zero curvature
- metric is convex if all points have zero or positive curvature
- metric is topological sphere if gluing noncrossing



shortest paths from x to all vxs.

Alexandrov's Theorem: [1941; English book 2005]

every convex polyhedral metric, topologically a sphere, is realized by a unique convex polyhedron
(possibly degenerating to doubly covered flat polygon)

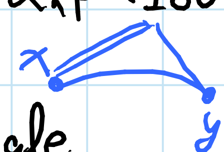


Proof sketch: [Lecture 14]

Uniqueness: draw all shortest paths between pairs of vxs.
- includes all edges of any polyhedral realization

$\Rightarrow$ faces between mesh of paths are rigid
- Cauchy's Rigidity Theorem $\Rightarrow$ unique convex realiz.

Existence: induct on $n = \#$vertices
- base case: $n \le 4$ (double triangle or tetrahedron)
- total curvature of all vertices $= 720° = 4\pi$
  [Descartes' Theorem; conseq. of Gauss-Bonnet Formula]
- $n \ge 5 \Rightarrow 2$ vertices x, y have curvatures $\alpha, \beta < 180°$
- along shortest path from x to y, paste edge of a doubly covered triangle



  $\Rightarrow$ new vertex @ triangle apex; adds material @ x & y
- continuously vary angles of triangle at x & y from $\emptyset$ to $\alpha/2$ & $\beta/2$ $\Rightarrow$ x & y flatten

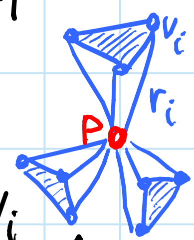
$\Rightarrow$ continuous path on manifold of metrics from original metric to metric with one less vertex
- induct on latter — lost x & y, gain apex
- argue continuity of realizability using Implicit Function Theorem $\Rightarrow$ nonconstructive $\square$

Constructive Alexandrov's Theorem: [Bobenko & Izmestiev 2006]

(following Blaschke & Herglotz 1937; Alexandrov 1950; Volkov 1955)

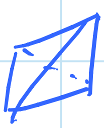
Idea: represent interior of polytope, not just boundary

- add (hypothetical) point p interior to polytope
- triangulate surface with geodesics
- form solid tetrahedron on p & each Δ
- solve for distance r_i from p to vertex v_i



$\Rightarrow$ determines geometry of tetrahedra, hence polytope

Generalized polytope: same combinatorial structure, tetrahedra glued around p, but not necc. in 3D



- consider dihedral angles of edges of tetrahedra ~ view as angle of solid material
- convexity invariant: Σ two dihedral angles incident to edge of surface triangulation $\leq 180°$
- goal: reach real polytope where $\kappa_i = 360° - \Sigma$ dihedral angles around interior edge $(p, v_i) = \emptyset$

Evolution: start at generalized polyhedron $P(\emptyset)$

- set $\kappa_i(t) = (1-t)\,\kappa_i(\emptyset) \to \emptyset$ as $t \to 1$
- differential equation to evolve r_i's:

$$\frac{d\vec{r}}{dt} = \underbrace{\left(\frac{\partial \vec{\kappa}}{\partial \vec{r}}\right)}_{\text{Jacobian - how } r_i\text{'s affect } \kappa_j\text{'s}}^{-1} \cdot \vec{r}(\emptyset)$$

- geodesic triangulation changes (flips) as $t \to 1$
- crucial part: Jacobian nonzero & has inverse (uses inverse function theorem!)

Constructive Alexandrov's Theorem: (cont'd)

Starting point: need a generalized polyhedron $P(\emptyset)$

1. geodesic Delaunay triangulation of surface
2. setting all r_i equal & sufficiently large yields desired convexity invariant
   - using Delaunay property

Pseudopolynomial algorithm for Alexandrov's Theorem:

[Kane, Price, Demaine 2009]

$O(n^{456.5} r^{1891} / \varepsilon^{121})$ time

ε → accuracy

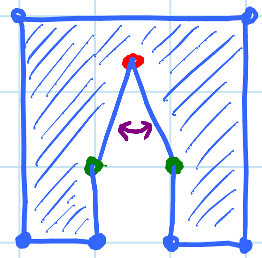
r → spread = largest dist./smallest dist.

- compute geodesic Delaunay by modifying [Mitchell, Mount, Papadimitriou 1987] to handle when edges not necc. shortest paths
- make each part effective with explicit bounds:
  - how large to make initial r_i's
  - Jacobian & inverse bounded away from $\emptyset$ (using Hessian instead of inverse function thm)

$\boxed{\text{OPEN}}$: polynomial time possible?

- logarithmic dependence on r/ε possible: reduces to roots of $2^{O(n)}$-degree polynomial [Sabitov 1996; Fedorchuk & Pak 2005]

Ungluable polygon: [Demaine, Demaine, Lubiw, O'Rourke 2000]



- no vertex can be glued into red reflex vertex: $< 90°$ free

$\Rightarrow$ "zip" red reflex vertex

$\Rightarrow$ green reflex vertices glued together

$\Rightarrow$ $> 360°$ of material $\square$

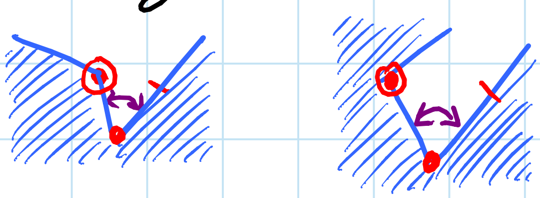
Random polygons are ungluable:

- suppose uniform distribution on angles & edge lengths

$\Rightarrow$ $\approx n/2$ reflex vertices

- gluing in a convex vertex still leaves reflex vertex (angles don't match)
- at some point must zip a reflex vertex
- fails if nearer angle is reflex:

convex $\Rightarrow$ OK



reflex $\Rightarrow$ BAD

- happens with probability $1/2$ for each reflex vertex $\square$

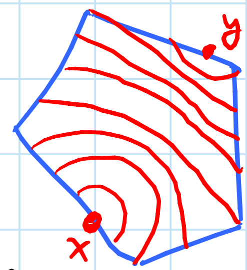
Perimeter halving: every convex polygon has an Alexandrov gluing

- pick any point x on polygon boundary
- glue together two boundary points at distance d from x (measured along boundary), for all $d > \emptyset$
  - both points have $\le 180°$ of material $\Rightarrow$ convex
- stop at diametrically opposite point y

$\Rightarrow$ gluing two halves (paths) of perimeter from x to y

- x & y also convex (nothing glued)

$\Rightarrow$ Alexandrov □



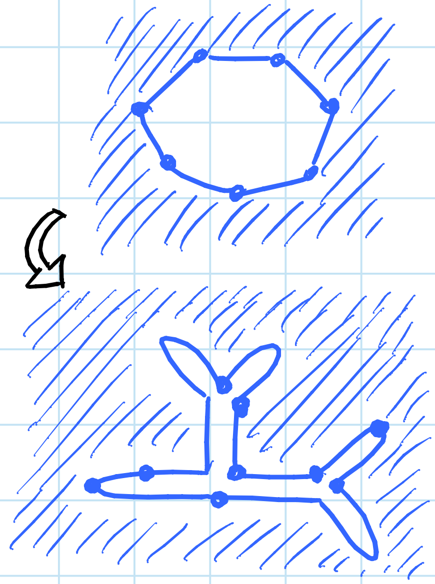
EXPERIMENT: cut out convex polygon
tape together perimeter halves
see what convex polyhedron you get

Mostly different: uncountably many polyhedra

- vary x near vertex v_i, say d along edge $v_i v_{i+1}$
- x & v_i become distinct vertices of shortest-path distance d
- only finitely many vertex-vertex shortest paths for a particular polyhedron
- uncountably many choices for d

$\Rightarrow$ uncountably many polyhedra □

<u>Gluing tree:</u>



- turn polygon "inside-out"
- gluing of that boundary to self forms a cycle around a tree

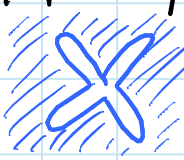
  gluing tree
- corresponds to cutting tree in unfolding

<u>Properties:</u>

- each leaf is either a zipped vertex or a fold point in middle of edge ($\Rightarrow 180°$)

  $\Rightarrow$ at most 4 fold points ($720°$ total curvature)
- if 4 fold points, then these are only leaves

  always induce curvature

  $\Rightarrow$  or 
- at most one nonvertex (middle of edge) glued at ≥ 3-way junction (else $180° \cdot 2$ + something)

Rolling belt = path in gluing tree whose end points are either fold pts. or convex vx. leaves & along which always $\le 180°$ material on either side

= effectively an embedded convex polygon

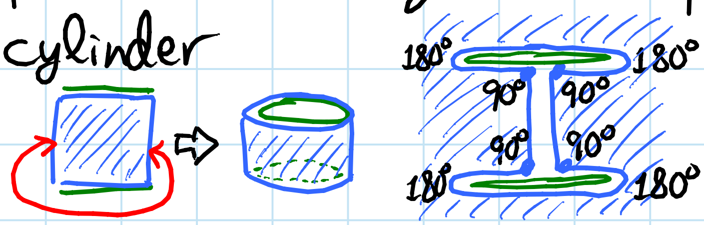
$\Rightarrow$ can perimeter halve arbitrarily = "rolling the belt"

— only way to get infinite gluings

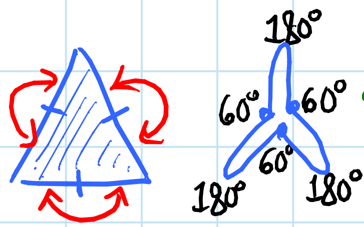
Examples:

1 rolling belt: perimeter halving of convex polygon

2 rolling belts: cylinder



3 rolling belts:



belt between every pair of leaves

≥4 rolling belts: impossible [6.885 Fall 2004 PS5.3]

— must be 4 fold points

$\Rightarrow$ no curvature elsewhere

$\Rightarrow$ rolling belt from one fold point is uniquely determined to some fold point

$\Rightarrow$ same rolling belt from latter fold point

$\Rightarrow \le 2$ rolling belts