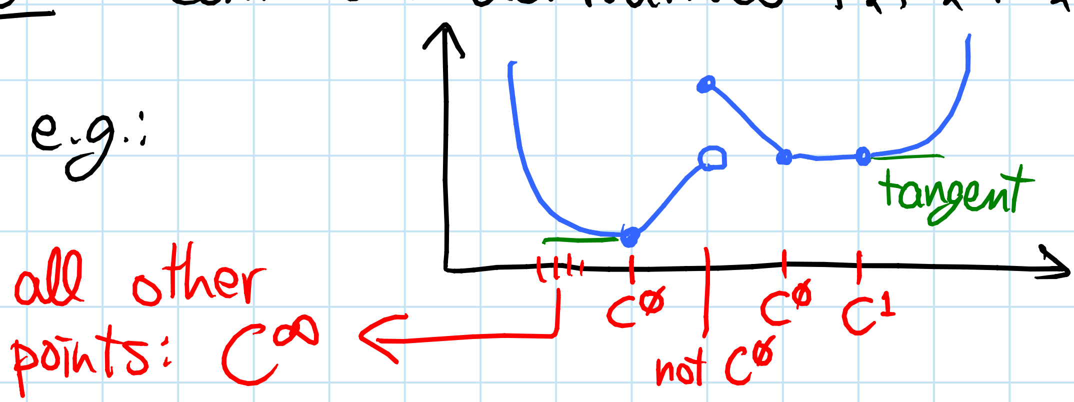
6.849	Class 9	Oct. 4, 2012

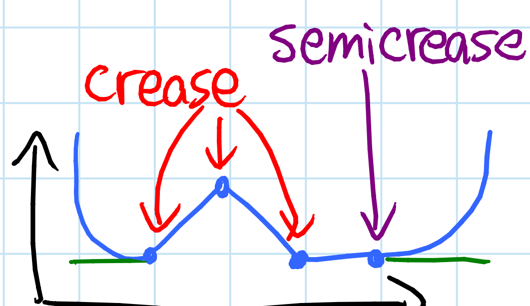
- <u>Triangulated hypars</u>:
  - **OPEN**: does alternating triangulation work for arbitrary n?
  - **OPEN**: do triangulations mixing alternation/not do medium well?
  - **OPEN**: what about regular k-gons?
    - <u>issue</u>: how to get started?

- <u>Smoothness</u>: → no gaps (undefined) & no jumps
  - <u>C^0</u> = continuous function f
  - <u>C^1</u> = continuous derivative f'
  - <u>C^2</u> = continuous second deriv. f''
  - <u>C^∞</u> = continuous derivatives $f, f', f'', \ldots$

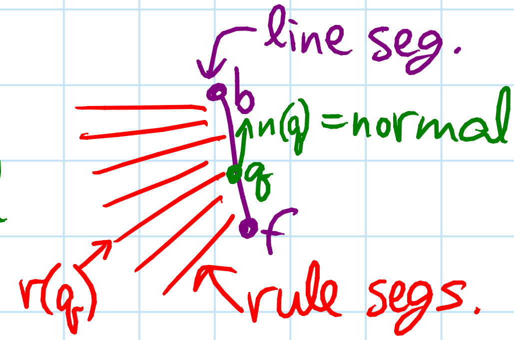
e.g.:



  - <u>well-behaved folding</u> = C^0 & piecewise-C^2



- **Polygonal ⇒ flat proof:**



  - $n(q) \perp bf$
  - $n(q) \perp r(q)$

  } definition of normal

  $\Rightarrow n'(q) \perp bf$

  $\&\ n'(q) \perp r(q)$

  … because $\underbrace{n(q+\varepsilon)}_{\text{move } q \text{ along } bf} \approx n(q) + \varepsilon \cdot n'(q)$ [Taylor series]

  $$\Rightarrow 0 = n(q+\varepsilon)\cdot bf \approx \underbrace{n(q)\cdot bf}_{=0} + \varepsilon \cdot \underbrace{n'(q)\cdot bf}_{=0} \quad \Rightarrow$$

  $$\& \ 0 = n(q+\varepsilon)\cdot r(q+\varepsilon)$$
  $$\approx \underbrace{n(q)\cdot r(q)}_{=0} + \varepsilon\,[\underbrace{n(q)\cdot r'(q)}_{=0} + \underbrace{n'(q)\cdot r(q)}_{=0}] + \varepsilon^2 \ldots \quad \Rightarrow$$

  coordinatize: point p on surface $\approx q + \varepsilon \cdot bf + \delta \cdot r(q+\varepsilon)$

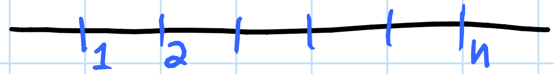
  <u>torsal</u> $\Rightarrow$ tangent plane $\perp n(q)$ $\Rightarrow r(q+\varepsilon) \perp n(q)$
  $\Rightarrow r'(q) \perp n(q)$

- **Mathematical vs. real paper:**

  what does it all mean?

  - real paper might stretch/shear slightly
  - real paper might add many small creases (too slight to see)

  $\boxed{\text{PROJECT}}$: investigate!

- Pleat folding algorithms:
  [Cardinal, Demaine, Demaine, Imahori, Ito, Kiyomi, Langerman, Uehara, Uno – G&C 2011]
  - model: 1D paper
    

    n uniformly spaced creases
    some-layers simple folds
    unfold previous fold (for free)
  - look at resulting string of M/V/unfolded
  - ≈ lg n folds makes "dragon curve":
    [Heighway, Banks, Harter ≈ 1967]
    M M V M M V M M V M V V M V V ...

    (& ≈ n variations)

  - MM⋯M can be folded by $\frac{3}{2}\lg^2 n$ folds
    1. fold in half until 3 creases left [xxx]
    2. fold them as M's; unfold all [MMM]
       → ?MMM?VVV?MMM?VVV?...
    3. fold VVV's on top of each other by rep. folding middle M of middle MMM
       → [M?VVV?M] → [MMMMMMM]
    4. fold 5 middle creases as M; unfold all
       → VM?$\underbrace{\text{MMMMMMM?MVVVVVM?}}$ etc.
    5. etc. $O(\lg n)$ rounds (doubling M reps.)
       - $O(\lg n)$ folds/round □

$\Rightarrow$ MVMV$\cdots$ can be folded by $3\lg^2\frac{n}{2}$ folds

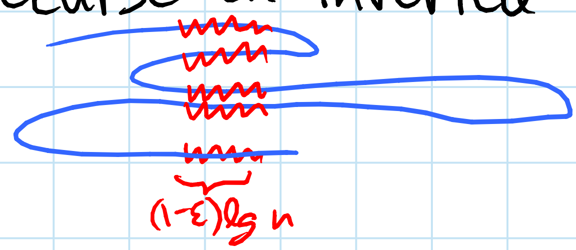
- MM$\cdots$M requires $\approx \frac{1}{4}\lg^2 n / \lg\lg n$ folds
  - let $f(n) = \#$ folds for $M^n \le \frac{3}{2}\lg^2 n$
  - $\Rightarrow$ fold $n/f(n)$ creases at once on average
  - $\Rightarrow$ <u>some</u> fold does $\ge n/f(n)$ creases at once
  - half of these creases become Valleys
  - need $\approx \lg n/f(n)$ folds to set this up
  - need $\approx \lg \frac{1}{2} n/f(n)$ folds to set up Mountain folding these valleys
  - etc.
  - $\Rightarrow$ need $f(n) \gtrsim \log_{\underbrace{f(n)}_{\le \frac{3}{2}\lg^2 n}} n \cdot \lg \frac{n}{\underbrace{f(n)}_{\le \frac{3}{2}\lg^2 n}} = \Omega\left(\frac{\lg^2 n}{\lg\lg n}\right)$ $\square$

$\boxed{\text{OPEN}}$: $\Theta\left(\frac{\lg^2 n}{\lg\lg n}\right)$ or $\Theta(\lg^2 n)$ optimal?

- MVMV$\cdots$ requires $\approx \frac{1}{4}\lg^2 n / \lg\lg n$ folds too

- most M/V strings require $\approx n/\lg n$ folds
  - k folds $\Rightarrow \le (2n)^k$ choices
    M/V where
  - another $\approx 4^k$ choices for how to place $\le k$ unfolds among k folds } Catalan
  - $k = \frac{n}{3+\lg n} \Rightarrow 4^k (2n)^k = (8n)^k = 2^{(3+\lg n)\cdot k} = 2^n$
  - 2^n M/V strings $\square$

- all M/V strings can be folded by $(4+\varepsilon)\, n/\lg n$
  - divide string into $\Theta(n/\lg n)$ chunks each of length $(1-\varepsilon)\lg n$
  - $2^{(1-\varepsilon)\lg n} = n^{1-\varepsilon}$ chunk values
    $\Rightarrow$ average chunk repeated $n^{\varepsilon}/\lg n$ times
  - for each chunk value:
    1. fold repetitions together
    2. make their $\Theta(\lg n)$ folds
    3. fix creases messed up in 1
    4. recurse on inverted half



$$n^{1-\varepsilon} \cdot \left(n^{\varepsilon}/\lg n + \lg n + n^{\varepsilon}/\lg n\right) \cdot 2$$

geometric series

$\boxed{\text{OPEN}}$: explicit (family of) examples requiring $\Theta(n/\lg n)$ folds?

$\boxed{\text{OPEN}}$: complexity of shortest fold sequence of given M/V string?
- known: in EXPTIME

- Let's fold hypars!