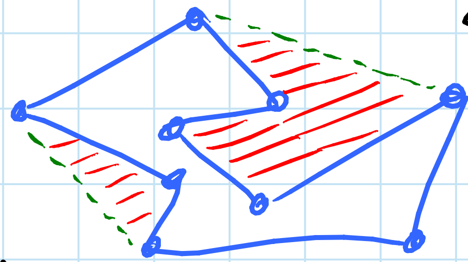
6.849	Lecture 21	Nov. 22, 2010

Pocket flipping: closed chains with universal joints are flat-state connected
(follows from Carpenter's Rule ~alternate approach)

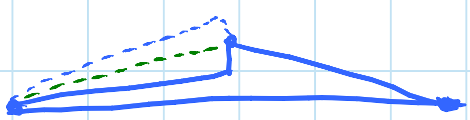
Pocket of 2D polygon = region outside polygon & inside convex hull



Pocket lid = convex-hull edge

Flip = reflect pocket through its lid
= rotate 180° through 3D around the lid
- avoids self-intersection (line of support)
- increases area

"Erdős-Nagy" Theorem: [posed by Erdős 1935]
any polygon always convexifies after finite flips, no matter how flip sequence is chosen
- but can be arbitrarily many:
[Joss & Shannon 1973]

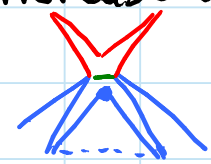


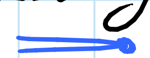
- $\boxed{\text{OPEN}}$: bound # flips in n & r = max. dist./min. dist
  - pseudopolynomial? [Overmars 1998]

"Proofs" of Erdős–Nagy Theorem: [Demaine, Gassend, O'Rourke, Toussaint 2007]

knowledge

- Nagy 1939 — <u>flawed</u>: "$P^0 \subseteq C^0 \subseteq P^1 \subseteq C^1 \subseteq \cdots$"
  (used to "prove" limit polygon convex)
- Reshetnyak 1957 — <u>correct</u> (though somewhat imprecise)
- Yusupov 1957 — <u>flawed</u>: "limit convex else flip" & more subtle error
- Bing & Kazarinoff 1959 — <u>correct</u> (though somewhat terse)
- Wegner 1993 — <u>flawed</u>: "move vertex ⇒ increase area by incident Δ"

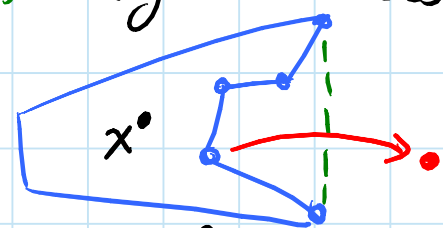


- all ← Grünbaum 1995 — <u>omission</u>: why limit polygon is convex
- all ← Toussaint 1999/2005 — <u>flawed</u>: "limit convex else flip"
- all ← Demaine, Gassend, O'Rourke, Toussaint 2008 — generalization to self-crossing assuming no "hairpins": 

Proof of "Erdős-Nagy" Theorem: [Bing & Kazarinoff 1959] & [DGOT 2006]

consider an infinite flip sequence

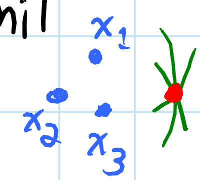
① distance from a vertex to fixed point x inside the polygon (remains so) only increases
- pocket lid is Voronoi diagram of old & new



② each vertex approaches a unique limit
- apply ① to three noncollinear points x_1, x_2, x_3 inside the polygon
- distances from vertex ≤ perimeter of polygon/2

⇒ distances converge

⇒ vertex approaches intersection of 3 circles



③ turn angle at each vertex converges
- by ②, 3 vertices defining the angle converge
- by ①, vertices do not get closer to each other
- rest by continuity

④ vertex moves infinitely ⇒ asymptotically flat
- each move negates sign of turn angle ⇒ → Ø

⑤ contradiction
- eventually asymptotically pointed vxs. stop moving

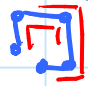
⇒ attain limit convex hull, but about to flip! □

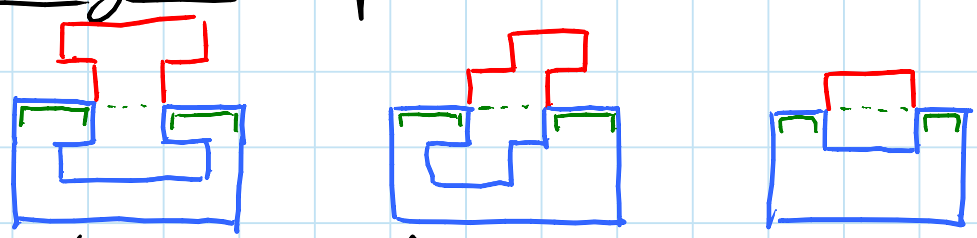
Flipturn: rotate pocket 180° in 2D around lid midpoint

- at most $n!$ configurations [Joss & Shannon 1973]
- always $O(n^2)$ flipturns [Aichholzer et al. 2002; Ahn et al. 2000 (diff. model)]
- sometimes $\Omega(n^2)$ flipturns [Biedl 2004]
- final polygon & location determined
- NP-hard to find longest flipturn sequence
- $\boxed{\text{OPEN}}$: finding shortest flipturn sequence?

[Aichholzer et al. 2002]

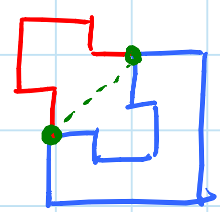
Orthogonal polygons: $< n$ flipturns

- count brackets:  &  (polygon interior on either side)
  - allow overlap  $\Rightarrow \le n$ brackets
- claim # brackets never decreases (13-case analysis)
- orthogonal flipturn kills two brackets:



$\Rightarrow \le n/2$ orthogonal flipturns

- diagonal flipturn kills two vertices:



$\Rightarrow < n/2$ diagonal flipturns

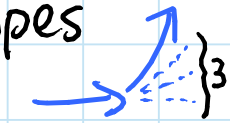
$\Rightarrow < n$ total $\square$

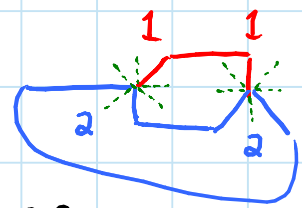
$\boxed{\text{OPEN}}$: $n - O(1)$ flipturns ever possible?

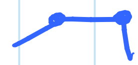
- best example requires $\frac{5}{6}n - O(1)$

Flipturns: (cont'd)

General polygons: $\le ns$ if s distinct slopes

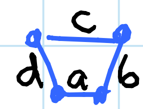
- discrete turn angle $= 1 + \#$slopes between 

- measure total discrete turn angle:
  - nondegenerate flipturn decreases by ≥ 2 

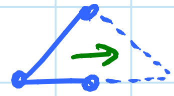
  - degenerate flipturn doesn't change
- also count brackets: 
  - nondegenerate flipturn increases by ≤ 2
  - degenerate flipturn decreases by ≥ 2
- potential function = total disc. angle $+ \frac{1}{2} \#$brackets
  - any flipturn decreases by ≥ 1
  - initially $\le n(s-1) + n = ns$ $\square$

Deflation: inverse of flip (avoiding crossings)

- conjectured finite [Wegner 1993]
- quadrilaterals with $a+c=b+d$ & $a\neq b\neq c\neq d\neq a$ always deflate infinitely [Fevens, Hernandez, Mesa, Morin, Soss, Toussaint 2001]

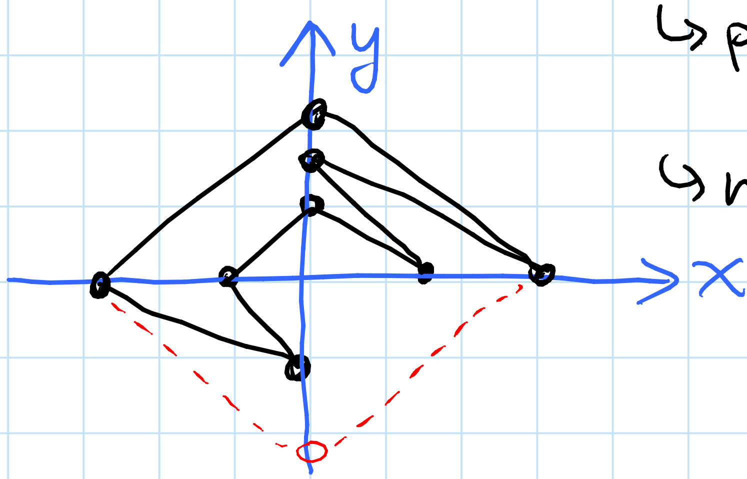


- that's all such quads [Ballinger, PhD 2003]
- no pentagon always deflates infinitely [Demaine, Demaine, Fevens, Mesa, Soss, Souvaine, Taslakian, Toussaint 2007]
- $\boxed{\text{OPEN}}$: any $n\geq 6$ gon with no flat vertices that always deflates to flat limit?
- any infinite deflation sequence has a unique limit polygon [Hubard & Taslakian 2010]
- $\boxed{\text{OPEN}}$: characterize infinitely deflatable polygons
  - algorithm for testing a sequence?

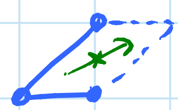
Pop: flip on 2 incident edges  [Millett 1994]

- can be forced to introduce crossings ⇒ allow
- possible to convexify any polygon in finitely many pops? [Ballinger & Thurston 2001]
  - NO: "alternating" polygons can't be [Dumitrescu & Hilscher 2009]

vertices alternate between x & y axes



↳ preserved under pops
↳ never convex for $n>4$

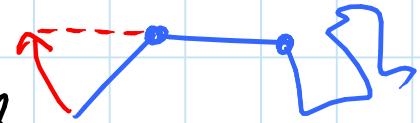
Popturns: flipturn on 2 incident edges 

- equivalent to pops for equilateral polygons
- can convexify any polygon allowing self-intersection
- characterization of when possible without crossing

[Aloupis et al. 2007]

Linkages in 4D: [Cocan & O'Rourke 2001]

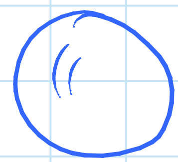
- every open chain can be straightened in 4D+:
  - idea: move first bar to "extend" second bar
  - then "fuse" that joint, treating first two bars as one



  - ⇒ effectively n−1 bars left: induct
  - problem: goal state for first bar might intersect rest of chain
    - if so, just perturb the linkage (actually can just move the vertex to be straightened)

works in 3D too!

  - key: first bar can reach any nonobstructed position
    - configurations around joint = points on 3D sphere in 4D (centered at joint)



      (analogy: 3D chain, points on usual 2D sphere)
    - obstacle = projection of 1D bar onto sphere = 1D arc
    - deleting 1D arcs keeps 3D sphere connected
      (analogy: deleting 0D points from usual 2D sphere)
- every tree can be flattened
  - similar technique
- every cycle can be convexified (diff. approach)