Solutions to Quiz 2

Problem 1 (40 points). *Hint:* Read both parts (a) and (b) before trying to solve either.

(a) (15 points) Give an example of an ⟨expression⟩, M, and a ⟨syntactic-value⟩, V, such that $M[x := \texttt{+}]$ is an ⟨immediate-redex⟩, but $M[x := V]$ is not.

Solution. Let $M ::= \texttt{(x 1 1)}, V ::= \texttt{list}$.

Many students confused an expression being an ⟨immediate-redex⟩, meaning it parses according to the grammar for ⟨immediate-redex⟩, and "matching the lefthand side of a Substitution Model rule," that is, being a *redex*. Unfortunately, the two notions are distinct, and neither even implies the other. ■

(b) (15 points) Suppose $M[x := \texttt{+}]$ is an ⟨immediate-redex⟩. Explain why $M[x := L]$ is also an ⟨immediate-redex⟩ for any ⟨lambda-expression⟩, L. (Don't get tangled in a complicated structural induction based on the BNF grammars for ⟨syntactic-value⟩ and ⟨immediate-redex⟩. Just explain clearly what properties(s) of the grammar(s) ensure the result.)

Solution. The only distinction among syntactic values in the grammar for ⟨immediate-redex⟩ is between ⟨syntactic-value⟩'s that are ⟨nonpairing-procedure⟩'s and those that are not. Since both + and L are ⟨nonpairing-procedure⟩'s, the ⟨immediate-redex⟩ grammar will parse $M[x := \texttt{+}]$ and $M[x := L]$ the same way, so if one is an ⟨immediate-redex⟩, the other one will be too. ■

(c) (10 points) Suppose

$$M[x := \texttt{+}] \rightarrow N[x := \texttt{+}].$$

Does it follow that

$$M[x := \texttt{(lambda (y) (+ y))}] \rightarrow N[x := \texttt{(lambda (y) (+ y))}]?$$

Prove or give a counterexample.

Solution. No, let $M ::= \texttt{(x 1)}, N ::= \texttt{1}$. So indeed,

$$M[x := \texttt{+}] = \texttt{(+ 1)} \rightarrow \texttt{1} = \texttt{1}[x := \texttt{+}] = N[x := \texttt{+}],$$

but

$$M[x := \texttt{(lambda (y) (+ y))}] \to \texttt{(letrec ((y 1)) (+ y))},$$

while

$$N[x := \texttt{(lambda (y) (+ y))}] = \texttt{1}.$$

Alternatively, let $M ::= \texttt{(x)}$, $N ::= \texttt{0}$. Now again

$$M[x := \texttt{+}] = \texttt{(+)} \to \texttt{0} = \texttt{0}[x := \texttt{+}] = N[x := \texttt{+}],$$

but

$$M[x := \texttt{(lambda (y) (+ y))}] \not\to$$

because it is an immediate error. ■

Problem 2 (30 points). Prove that observational equivalence, $\equiv$, (the definition is in an appendix in case you don't remember it) is a congruence relation on Scheme expressions, that is,

1. it is a reflexive, symmetric, and transitive relation, and
2. $M \equiv N$ implies $C[M] \equiv C[N]$ for any context, C.

Solution. *Proof.* 1. • [reflexive] M is trivially indistinguishable from M, hence $M \equiv M$ by definition.

- [symmetry] We prove the contrapositive: if M is distinguishable from N, then N is distinguishable from M. But the definition of distinguishability is symmetric in M and N, so this follows immediately.
- [transitivity] Suppose $L \equiv M$ and $M \equiv N$. We want to show that $L \equiv N$. So suppose to the contrary, that L & N were distinguishable, that is, there is a context, C, such that

$$C[L]\downarrow \quad \text{iff} \quad C[N]\not\downarrow. \tag{1}$$

But,

$$C[L]\downarrow \quad \text{iff} \quad C[M]\downarrow \tag{2}$$

since $L \equiv M$, and likewise

$$C[M]\downarrow \quad \text{iff} \quad C[N]\downarrow. \tag{3}$$

Combining (1), (2), (3), we have

$$C[N]\not\downarrow \quad \text{iff} \quad C[L]\downarrow \quad \text{iff} \quad C[M]\downarrow \quad \text{iff} \quad C[N]\downarrow,$$

a contradiction.

2. We prove the contrapostive.

   Assume that $C[M]$ and $C[N]$ are distinguishable. So there is a context, D, such that

   $$D[C[M]]\downarrow \quad \text{iff} \quad D[C[M]]\not\downarrow.$$

   So $D[C]$ is a distinguishing context for M and N, proving that M and N are distinguishable.

□

■

Problem 3 (30 points). Prove that if $M =_\alpha N$, then $M \equiv N$. (You may cite any of the facts in Notes 7.)

Solution. We know that $M =_\alpha N$ and implies $C[M] =_\alpha C[N]$ for any context, C. Also, the Substitution Model rules preserve $=_\alpha$. So it follows that if $C[M]$ rewrites to K in n applications of rules, then $C[N]$ rewrites in n rule applications to some K' such that $K =_\alpha K'$. Moreover, the grammar for ⟨syntactic-value⟩'s and hence for final values does not depend on the names of variables, so being a final value is also preserved by $=_\alpha$, that is, K is a final value iff K' is a final value. Hence if $C[M] \downarrow K$, then $C[N]\downarrow K'$, which shows that C is not a distinguishing context for M and N. Since this holds for any C, it follows that M and N are indistinguishable. ■

A Observational Equivalence

Definition 3.1. Two Scheme expressions, M and N, are said to be *observationally distinguishable* iff there is a context, C, such that exactly one of $C[M]$ and $C[N]$ converges. Such a context is called a *distinguishing context* for M and N. If M and N are not observationally distinguishable, they are said to be *observationally equivalent*, written $M \equiv N$.