Appendix

1 The Meanings of Terms

Definition. A *model*, $\mathcal{M}$, for signature, Σ, consists a nonempty set, $\mathcal{D}_{\mathcal{M}}$, called the *domain* of $\mathcal{M}$, and a mapping that assigns an n-ary operation on the domain to each symbol of arity n in Σ. That is, letting $[\![f]\!]_0$ be the meaning of $f \in \Sigma$, we have for each f of arity $n > 0$,

$$[\![f]\!]_0 : (\mathcal{D}_{\mathcal{M}})^n \to \mathcal{D}_{\mathcal{M}},$$

and for each $c \in \Sigma$ of arity 0,

$$[\![c]\!]_0 \in \mathcal{D}_{\mathcal{M}}.$$

An *$\mathcal{M}$-valuation*, V, is a mapping from variables into the domain, $\mathcal{D}_{\mathcal{M}}$.

The meaning, $[\![M]\!]$, of term, M, in model, $\mathcal{M}$, is a function from valuations to values in the domain. It is defined by structural induction on the definition of M:

$$\begin{aligned} [\![x]\!]V &::= V(x) && \text{for each variable, } x,. \\ [\![c]\!]V &::= [\![c]\!]_0 && \text{for each constant, } c \in \Sigma, \\ [\![f(M_1,\dots,M_n)]\!]V &::= [\![f]\!]_0([\![M_1]\!]V,\dots,[\![M_n]\!]V) && \text{for each } f \in \Sigma \text{ of arity } n > 0. \end{aligned}$$

1.1 The Meanings of Applications

Definition. For any function, F, and elements a, b, we define $F[a \leftarrow b]$ to be the function G such that

$$G(u) = \begin{cases} b & \text{if } u = a. \\ F(u) & \text{otherwise.} \end{cases}$$

We can extend the set of terms to allow *applications* of the form $((\lambda(x)M)\ N)$ whose meaning is defined by the rule

$$[\![((\lambda(x)M)\ N)]\!]V ::= [\![M]\!](V[x \leftarrow [\![N]\!]V]).$$

2 Validity of Equations

Definition. An equation is *valid* in a model $\mathcal{M}$, written,

$$\mathcal{M} \models M = N,$$

iff $[\![M]\!]_{\mathcal{M}} = [\![N]\!]_{\mathcal{M}}$. When $\mathcal{E}$ and $\mathcal{E}'$ are sets of equations, we write

$$\mathcal{M} \models \mathcal{E}$$

to mean that $\mathcal{M} \models M = N$ for each equation $(M = N) \in \mathcal{E}$. We write

$$\mathcal{E} \models \mathcal{E}'$$

to mean that

$$\mathcal{M} \models \mathcal{E} \quad \text{implies} \quad \mathcal{M} \models \mathcal{E}'$$

for every model, $\mathcal{M}$.

3 Substitution

Definition. A *substitution* is a mapping, σ, from a set of variables to terms. The notation

$$[x_1, \ldots, x_n := M_1, \ldots, M_n]$$

describes the substitution that maps variables $x_1, \ldots, x_n$ respectively to terms $M_1, \ldots, M_n$, and maps all other variables to themselves.

Every substitution, σ, defines a mapping, $[\sigma]$, from terms to terms defined inductively as follows:

$$\begin{aligned}
c[\sigma] &::= c && \text{for each constant, } c,\\
x[\sigma] &::= \sigma(x) && \text{for each variable, } x,\\
f(M_1, \ldots, M_n)[\sigma] &::= f(M_1[\sigma], \ldots, M_n[\sigma]) && \text{for each } f \in \Sigma \text{ of arity } n > 0,\\
((\lambda(x)M)\ N)[\sigma] &::= ((\lambda(x')M[\sigma'])\ N[\sigma]) && \text{where } x' \text{ is fresh, and } \sigma' ::= \sigma[x \leftarrow x'].
\end{aligned}$$

Lemma (General Substitution). *Let σ be a substitution, V a valuation, and V_σ be the valuation such that*

$$V_\sigma(x) ::= [\![\sigma(x)]\!]V$$

for all variables, x. Then for every term, M,

$$[\![M[\sigma]]\!]V = [\![M]\!]V_\sigma,$$

4 Proofs

Definition. A *sequence-of-equations proof* is a finite sequence of *equations* such that every equation in the sequence follows from equations earlier in the sequence by one of the standard equational inference rules, starting from a given set of equational axioms.

An *substitution proof* is a sequence of *terms*,

$$M_0, M_1, \ldots, M_n$$

such that M_{i+1} is the result of replacing a subterm, L_i, of M_i by a term, K_i, where $K_i = L_i$ or $L_i = K_i$ is a substitution instance of an axiom, for $i = 1, \ldots, n$.

5 Soundness & Completeness

Theorem (Axiomatic Completeness). *$\mathcal{E} \models M = N$ iff $M = N$ is provable using the rules of Table 1.*

Theorem (Arithmetic Completeness). *An arithmetic equation $e = f$ is valid over the reals iff it is valid over the integers iff it is provable using the rules of Tables 1 and 2.*

Theorem (Arithmetic Soundness). *If an arithmetic equation or inequality is provable using the rules of Tables 1, 2, and 3, then it is valid over the integers.*

Table 1: Standard Equational Inference Rules.

	$\Longrightarrow$	$M[\sigma]$	$= N[\sigma]$ for $(M = N) \in \mathcal{E}$	($\mathcal{E}$ substitution)
	$\Longrightarrow$	M	$= M$	(reflexivity)
$M = N$	$\Longrightarrow$	N	$= M$	(symmetry)
$L = M, M = N$	$\Longrightarrow$	L	$= N$	(transitivity)
$M_1 = N_1, \ldots, M_n = N_n$	$\Longrightarrow$	$f(M_1, \ldots, M_n)$	$= f(N_1, \ldots, N_n)$ when $\text{arity}(f) = n > 0$	(congruence)

Table 2: Equational Axioms for Arithmetic

$(e+f)+g$	$=$	$e+(f+g)$	(associativity of $+$)
$(e \cdot f) \cdot g$	$=$	$e \cdot (f \cdot g)$	(associativity of $\cdot$)
$e+f$	$=$	$f+e$	(commutativity of $+$)
$e \cdot f$	$=$	$f \cdot e$	(commutativity of $\cdot$)
$0+e$	$=$	e	(identity for $+$)
$1 \cdot e$	$=$	e	(identity for $\cdot$)
$e+(-e)$	$=$	0	(inverse for $+$)
$e \cdot (f+g)$	$=$	$(e \cdot f)+(e \cdot g)$	(distributivity)

Table 3: Inference Rules for Inequalities.

$e=f$	$\Longrightarrow$	$e \leq f$	($\leq$-reflexivity)
$e \leq f,\ f \leq e$	$\Longrightarrow$	$f=e$	($\leq$-antisymmetry)
$e \leq f,\ f \leq g$	$\Longrightarrow$	$e \leq g$	($\leq$-transitivity)
$e_1 \leq e_2,\ f_1 \leq f_2$	$\Longrightarrow$	$e_1+f_1 \leq e_2+f_2$	($+$-$\leq$-congruence)
$e_1 \leq e_2,\ 0 \leq f_1 \leq f_2$	$\Longrightarrow$	$e_1 \cdot f_1 \leq e_2 \cdot f_2$	($\cdot$-$\leq$-congruence)
$e \leq f$	$\Longrightarrow$	$-f \leq -e$	($-$-$\leq$-congruence)
	$\Longrightarrow$	$0 \leq 1$	(01-axiom)