

Solutions to Assignment 3.5

Problem 1. Extend Arithmetic Expressions with another case called an application. Namely, if e and f are ae's and x is a variable, and then the *application of e , regarded as a function of x , to f* , written

$$((\lambda(x)e)f),$$

is also an ae. The meaning of the application is given by the rule

$$\llbracket ((\lambda(x)e)f) \rrbracket V ::= \llbracket e \rrbracket (V[x \leftarrow \llbracket f \rrbracket V]) \quad (1)$$

(a) Extend the definition of the substitution operation, $[\sigma]$, so it applies to ae's with applications and so that the Substitution Lemma will continue to hold.

Hint: Substitution into an application will now generally have to include renaming lambda-bound identifiers to avoid "false capture." For example,

$$\begin{aligned} \llbracket ((\lambda(y) ((\lambda(x)(x + y)) 0)) x) \rrbracket V &::= \llbracket ((\lambda(x)(x + y)) 0) \rrbracket V[y \leftarrow V(x)] \\ &= \llbracket x + y \rrbracket ((V[y \leftarrow V(x)])[x \leftarrow 0]) \\ &= V(x), \end{aligned}$$

whereas if we try evaluating by naively substitute x for y inside the inner application, we get a different result:

$$\begin{aligned} \llbracket ((\lambda(x)((x + y)[y := x])) 0) \rrbracket V &= \llbracket ((\lambda(x)2x) 0) \rrbracket V \\ &= 0. \end{aligned}$$

The fix is to replace the x 's in the inner application by a "fresh" variable, x' before substituting x for y :

$$\begin{aligned} \llbracket ((\lambda(x')(x' + y)[y := x]) 0) \rrbracket V &= \llbracket ((\lambda(x')(x' + x)) 0) \rrbracket V \\ &= \llbracket x' + x \rrbracket (V[x' \leftarrow 0]) \\ &= V(x). \end{aligned}$$

Solution. Substitution into an application will be defined by the rule:

$$((\lambda(x)e)f)[\sigma] ::= ((\lambda(x')e[\sigma'])f[\sigma]) \quad (2)$$

where x' is a “fresh” variable — a variable not occurring in e , f , or σ — and

$$\sigma' ::= \sigma[x \leftarrow x'].$$

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(b) Prove it.

Solution.

Lemma (General Substitution).

$$\llbracket M[\sigma] \rrbracket V = \llbracket M \rrbracket V_\sigma, \quad (3)$$

where

$$V_\sigma(y) ::= \llbracket \sigma(y) \rrbracket V.$$

The proof by structural induction on M of the Substitution Lemma for ordinary ae’s carries over without change for ae’s with applications, except that there is now one further induction case, namely when M is an application, $((\lambda(x)e)f)$. In this case, the lefthand side of (3) is

$$\begin{aligned} \llbracket ((\lambda(x)e)f)[\sigma] \rrbracket V &= \llbracket ((\lambda(x')e[\sigma'])f[\sigma]) \rrbracket V && \text{(by (2))} \\ &= \llbracket e[\sigma'] \rrbracket U && \text{(by (1))} \\ &\quad \text{where } U ::= V[x' \leftarrow \llbracket f[\sigma] \rrbracket V] \\ &= \llbracket e \rrbracket U_{\sigma'} && \text{(ind. hypothesis for } e) \end{aligned} \quad (4)$$

Similarly, the righthand side of (3) is

$$\begin{aligned} \llbracket ((\lambda(x)e)f) \rrbracket V_\sigma &= \llbracket e \rrbracket (V_\sigma[x \leftarrow \llbracket f \rrbracket V_\sigma]) && \text{(by (1))} \\ &= \llbracket e \rrbracket W && (5) \\ &\quad \text{where } W ::= V_\sigma[x \leftarrow \llbracket f \rrbracket V_\sigma]. && (6) \end{aligned}$$

So we need only prove that

$$U_{\sigma'} = W \quad (7)$$

to conclude from (4) and (5) that the General Substitution Lemma holds for an application.

To prove (7), we use a key property of “fresh” variables:

Lemma (Fresh Variable). For any ae, M , valuation, V , and value, d , if x' does not occur in M , then

$$\llbracket M \rrbracket V = \llbracket M \rrbracket (V[x' \leftarrow d])$$

The proof of the Fresh Variable Lemma follows immediately by structural induction on M , as the reader can easily verify.

Now for a variable y distinct from x , we have

$$\begin{aligned}
 U_{\sigma'}(y) &= \llbracket \sigma'(y) \rrbracket U && \text{(by def. of (valuation)}_{\sigma'}) \\
 &= \llbracket \sigma(y) \rrbracket U && \text{(def. of } \sigma') \\
 &= \llbracket \sigma(y) \rrbracket V && \text{(by Fresh Var. Lemma since } x' \notin \sigma(y)) \\
 &= V_{\sigma}(y) && \text{(def. of } V_{\sigma}) \\
 &= W(y) && \text{(since } y \text{ is not } x)
 \end{aligned}$$

Also,

$$\begin{aligned}
 U_{\sigma'}(x) &= \llbracket \sigma'(x) \rrbracket U && \text{(by def. of (valuation)}_{\sigma'}) \\
 &= \llbracket x' \rrbracket U && \text{(def. of } \sigma') \\
 &= \llbracket f[\sigma] \rrbracket V && \text{(def. of } U) \\
 &= \llbracket f \rrbracket V_{\sigma} && \text{(ind. hypothesis for } f) \\
 &= W(x) && \text{(def. of } W)
 \end{aligned}$$

So we have shown that $U_{\sigma'} = W$, completing the proof. ■