6.841 Advanced Complexity Theory April 1, 2009

Lecture 15

Lecturer: Madhu Sudan *Scribe: Rotem Oshman*

1 Today's topics

- Private coins $\equiv$ public coins (that is, IP$[k] \equiv$ AM)
  - Goldwasser-Sipser approximate counting protocol
- Towards protocols for the Permanent, in the goal of showing that #P $\subseteq$ IP.

2 Review of last lecture

2.1 Graph Non-Isomorphism

The graph non-isomorphism problem is to decide the language GNI $= \{(G_0, G_1) \mid G_0 \not\sim G_1\}$. Last lecture we saw a private-coin 2-round protocol for GNI:

- The Verifier chooses a permutation $\pi \in_R S_n$, where n is the number of nodes in the graph, and a bit $b \in_R \{0, 1\}$. It sends $H = \pi(G_b)$ to the Prover.
- The Prover returns with a bit b'.
- The Verifier accepts if $b = b'$.

If $G_0 \sim G_1$, then the bit b is independent of H, which means that the Prover is essentially guessing b' and has a probability of $1/2$ of being correct. If $G_0 \not\sim G_1$, then H identifies b uniquely and the Prover will always be correct.

2.2 Kilian's protocol: IP$[k] \subseteq$ AM[poly]

Last time we saw a public-coin protocol through which the Prover could convince the Verifier that there are "many" coin tosses that would have made the Verifier accept in the original *private-coin* protocol. Kilian's protocol is public coin and has completeness of 1, but the number of rounds in the protocol depends on the number of random coins used in the original private-coin protocol. For example, the public-coin protocol we would get for GNI would have $O(n \log n)$ rounds instead of 2.

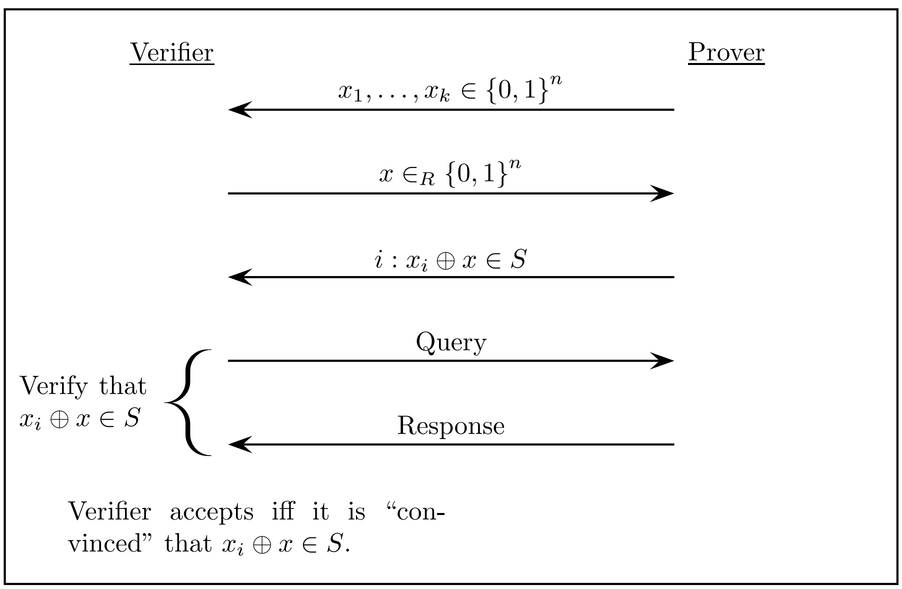
3 The Goldwasser-Sipser Protocol

Let $S \subseteq \{0,1\}^n$ be a set, such that membership in S is verifiable in AM. We are interested in solving the promise problem given by

$$\Pi_{\text{YES}} = \{S : |S| \geq f(n)\}$$
$$\Pi_{\text{NO}} = \left\{S : |S| < \frac{f(n)}{10n^2}\right\}$$

3.1 Initial attempt

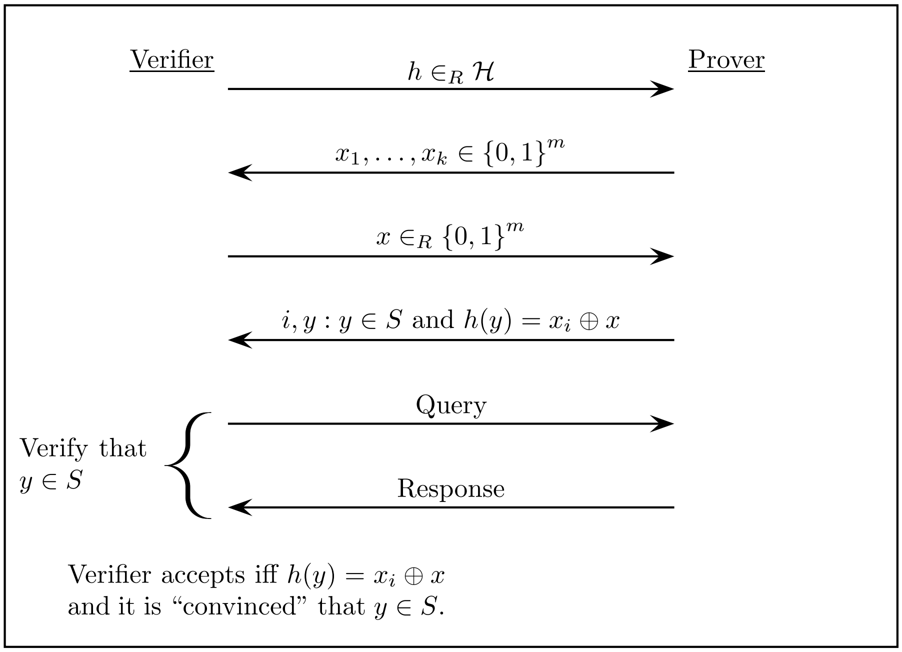
Suppose that $f(n)$ is "very large": $f(n) \approx 2^n$ (actually a little less than that). In this case, for YES instances we have $\frac{|S|}{|\{0,1\}^n|} \approx 1$, and for NO instances we have $\frac{|S|}{|\{0,1\}^n|} \lesssim \frac{1}{n^2}$. In other words, when we choose a random string, for YES instances our chance of hitting a member of S is very high, and for NO instances it is about 1 in n^2. We can re-use ideas from the proof that $\text{BPP} \subseteq \Sigma_2^{\text{P}}$ to convince the Verifier that $|S| \geq f(n)$. The protocol would be as follows.



Unfortunately, this only works when $f(n) \approx \frac{2^n}{\text{poly}}$.

3.2 Using hasing to overcome small $f(n)$'s

We can use pairwise-independent hashing to map S to a smaller space $\{0,1\}^m$, so that if S is large, then $h(S)$ will be a very large fraction of $\{0,1\}^m$. If h is a hash function that has no collisions in S then $|h(S)| = |S|$; therefore, if we choose $m \approx \log f(n)$ and S is a YES instance ($|S| \geq f(n)$), then in the hash-space we will have $|h(S)| = |S| \approx 2^m$. Then we can execute the previous protocol in the hash-space:



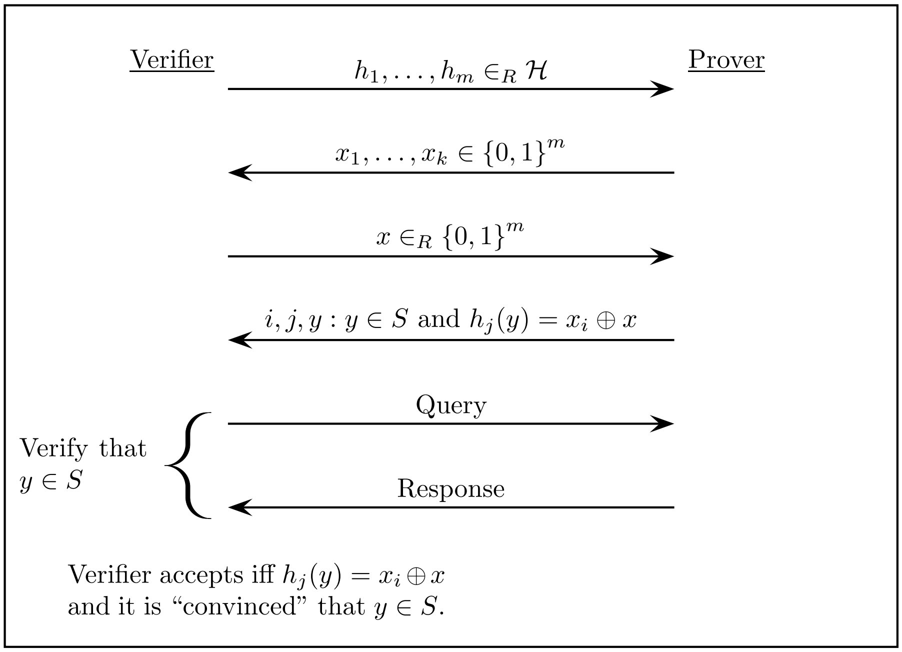
where $\mathcal{H}$ is a pairwise-independent family of hash functions.

The problem with *this* is that $h(S)$ is only guaranteed to be large if h has no (or few) collisions in S; if $h(S)$ is too small we have the same problem we had when $f(n)$ was small — the prover may not have a good response $x_1, \ldots, x_k$ in the second round.

To decrease the chances of this bad event we will use more than one hash function, so that the probability that *one* of the hash functions is good will be very high. We need to make sure that if S is small (a NO instance), our multiple hash functions will not give the Prover too much freedom by mapping S to a large fraction of $\{0,1\}^m$.

3.3 Final protocol

Let $m = \log f(n) + 2$. The protocol is as follows.



3.3.1 Analysis

First suppose that S is a NO instance, that is, $|S| < \frac{f(n)}{10n^2}$. Then

$$\frac{|\bigcup_j h_j(S)|}{|\{0,1\}^m|} \leq \frac{m \cdot |S|}{2^m} < \frac{(\log f(n) + 2) \cdot f(n)}{10n^2 \cdot 4f(n)} \leq \frac{1}{20n}$$

(In the last step we used the fact that $2 \leq f(n) \leq 2^n$, otherwise the problem definition does not make sense). It follows that for NO instances, for every choice of $x_1, \ldots, x_n$ that the Prover may send, the Verifier has a high probability of selecting a string x such that for all i, $x_i \oplus x \notin \bigcup_j h_j(S)$. The Prover will not have a response i, j, y that has a high probability of making the Verifier accept.

Now suppose that S is a YES instance, that is, $|S| \geq f(n)$. We are interested in the probability of choosing $h_1, \ldots, h_m$ such that for some $j \in \{1, \ldots, m\}$ we have $|h_j(S)| \geq f(n)$. This ensures that the Prover has a response in the second round.

Since each $\mathcal{H}$ is pairwise-independent, for any fixed $x \neq y \in \{0,1\}^n$ and for every $a, b \in \{0,1\}^m$ we have

$$\Pr_{h \in \mathcal{H}}[h(x) = a \wedge h(y) = b] = \frac{1}{4^m}$$

and hence,

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \sum_{a \in \{0,1\}^m} \Pr_{h \in \mathcal{H}}[h(x) = a \wedge h(y) = a] \leq 2^m \cdot \frac{1}{4^m} = \frac{1}{2^m}$$

Let $S' \subseteq S$ be some subset of size $f(n)$ of S (recall that $|S| \geq f(n)$). By a union bound over elements of S' ("unfixing" y) we obtain

$$\Pr_{h \in \mathcal{H}}[\exists y \in S' : x \neq y \wedge h(x) = h(y)] \leq \frac{f(n)}{2^m} = \frac{f(n)}{4f(n)} = \frac{1}{4}$$

Therefore,

$$\Pr_{h_1,\ldots,h_m \in \mathcal{H}}[\forall j \exists y \in S' : x \neq y \wedge h_j(x) = h_j(y)] \leq \frac{1}{4^m}$$

Finally, by another union bound over the elements of S' ("unfixing" x this time),

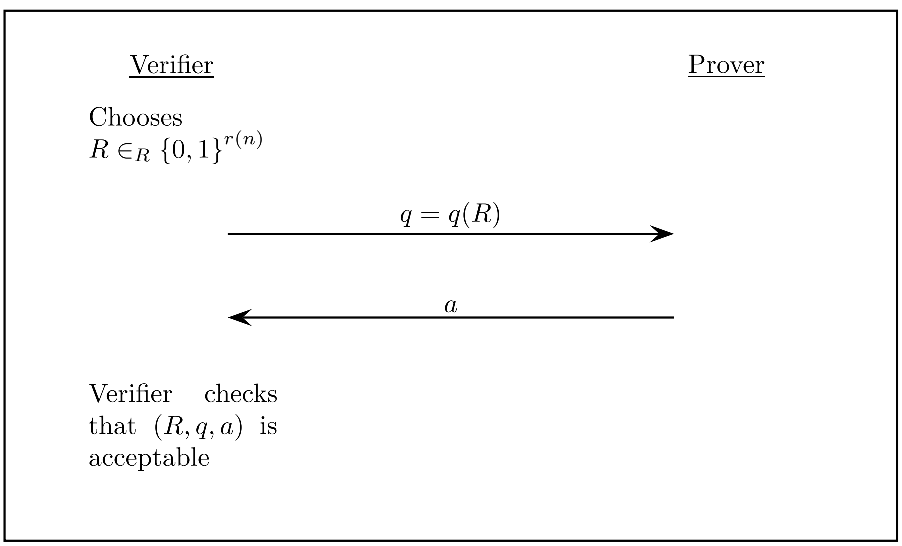
$$\Pr_{h_1,\ldots,h_m \in \mathcal{H}}[\exists x \in S' \forall j \exists y \in S' : x \neq y \wedge h_j(x) = h_j(y)] \leq \frac{f(n)}{4^m} = \frac{f(n)}{8f(n)} < \frac{1}{f(n)}$$

It follows that with probability all but $1/f(n)$ there will be at least one h_j that maps $f(n)$ elements of S to distinct elements of $\{0,1\}^m$, which implies $|\bigcup_j h_j(S)| \geq f(n)$. This is sufficient to use the $\mathrm{BPP} \subseteq \Sigma_2^{\mathrm{P}}$-style protocol above.

4 $\mathrm{IP}[2] \subseteq \mathrm{AM}$

We will use the Goldwasser-Sipser approximate counting protocol to convert a 2-round private coin protocol into a constant-round public-coin protocol. Since $\mathrm{AM}[k] = \mathrm{AM}$ for any constant k (see problem set #3), this will show that $\mathrm{IP}[2] \subseteq \mathrm{AM}$.

A 2-round private-coin protocol looks like this:



Define $S_{q,a} = \{R \mid (R, q, a) \text{ is acceptable}\}$. If we make the simplifying assumption that there is some number N such that

1. In YES instances, $\forall q \exists a : |S_{q,a}| \geq N$, and
2. In NO instances, $\forall q \forall a : |S_{q,a}| < \frac{N}{10n^2}$, and

3. N is *known to the Verifier*,

then the Prover could convince the Verifier that it should accept by sending a pair (q, a) that it claims has $|S_{q,a}| \geq N$, and then the Verifier and the Prover would run the GS protocol to verify that $|S_{q,a}| \geq N$. However, in reality, N is not known to the verifier, and furthermore N can depend on q (that is, $N = N(q)$).

The solution is to have the Prover provide a number N in the first round that it claims satisfies $|Q_N| \geq \frac{2}{3}\varepsilon \cdot \frac{2^n}{N}$, where ε is some constant and where $Q_N = \{q \mid \exists a : |S_{q,a}| > N\}$. That is, the prover provides a number N such that "many" queries q have an answer a with $|S_{q,a}| > N$. Then the Verifier and Prover execute the GS protocol to verify the size of Q_N. After this, N is used in the protocol from before.

5 Properties of the Permanent

Recall that $\text{perm}(A) = \sum_\sigma \prod_{i=1}^n A_{i,\sigma(i)}$, where $A \in \mathbb{Z}_p^{n\times n}$.

- **Random self-reducibility:** for fixed $A, R \in \mathbb{Z}_p^{n\times n}$, let $p(i) = \text{perm}(A + i \cdot R)$. This is a degree-$n$ polynomial in i. For $i = 0$ we have $p(i) = \text{perm}(A)$.

  Suppose that we have an algorithm M that computes $\text{perm}(R)$ with probability $1 - \varepsilon$ for random matrices R. Since $\mathbb{Z}_p$ is a field, for $i \neq 0$ and randomly chosen R, $A + i \cdot R$ is also distributed uniformly. Therefore,

  $$\Pr\left[M(A + i \cdot R) = \text{perm}(A + i \cdot R)\right] \geq 1 - \varepsilon.$$

  It follows that

  $$\Pr\left[\forall i \in \{1, \ldots, n\} : M(A + i \cdot R) = \text{perm}(A + i \cdot R)\right] \geq 1 - n \cdot \varepsilon,$$

  that is, we can compute w.h.p. the values $p(1), \ldots, p(n)$. These values can be used to interpolate p (which is of degree n) and compute $p(0) = \text{perm}(A)$.

- **Downward self-reducibility:** suppose we have an algorithm M that computes $\text{perm}(B)$ for $B \in \mathbb{Z}_p^{n-1\times n-1}$ (a smaller matrix). For an n-by-n matrix A, we can write $\text{perm}(A) = \sum_{i=1}^n (a_{1,i} \cdot \text{perm}(A \setminus i))$, where $A \setminus i$ is the matrix obtained from A by removing the first row and the i-th column. Thus, we can compute $\text{perm}(A)$ using n calls to $M(B)$, and if M terminates in polynomial time then so will the computation of $\text{perm}(A)$.

These two properties are used in an alternating manner in an interactive proof for the Permanent problem.