Advanced Complexity Theory
6.841/18.405J - Spring 2009
Due: Friday, April 03, 2009

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Problem Set 3

Problems

1. **Promise-RP and Promise-BPP**

   (a) Define the promise classes Promise-RP (prRP) and Promise-BPP (prBPP) so that they have complete (promise) problems, and so that they include RP and BPP (respectively).

   (b) Define $\mathrm{prRP}^{\mathrm{prRP}}$ and $\mathrm{prBPP}^{\mathrm{prBPP}}$.

   (c) Prove that $\mathrm{prBPP}^{\mathrm{prBPP}} \subset \mathrm{prBPP}$.

   (d) Prove that $\mathrm{prRP}^{\mathrm{prRP}} = \mathrm{prBPP}$. *Hint:* Review the proof of $\mathrm{BPP} \subset \Sigma^2$.

   (e) Conclude that if $\mathrm{P} = \mathrm{prRP}$, then $\mathrm{P} = \mathrm{prBPP}$.

2. **The hierarchy does not have fixed-polynomial size circuits**
   Prove that for any fixed k, $\mathrm{TIME}(n^k)/n^k$ (or roughly equivalently $\mathrm{SIZE}(n^k)$, the class of functions computed by circuits of size n^k) does not contain the Polynomial Hierarchy. (*More* credit for showing that *lower* levels of the hierarchy are not contained in $\mathrm{TIME}(n^k)/n^k$. [**Corrected 3/17/09**.]

3. PH **vs.** PSPACE
   Prove that there is an oracle A such that $\mathrm{PH}^A \neq \mathrm{PSPACE}^A$. *Hint:* Parity is not in AC^0.

4. **Sparse** NP**-complete languages**
   A language L is said to be *sparse* if there exists a k such that for all sufficiently large n, $|L \cap \{0,1\}^n| < n^k$. Prove that if a sparse language is NP-complete, then the Polynomial Hierarchy collapses. (*More* credit for collapsing it to *lower* levels).

5. $\mathrm{AM}[2] = \mathrm{AM}$
   Write out the explicit definition of the complexity classes $\mathrm{BP} \cdot \exists \cdot \mathrm{P}$ and $\mathrm{BP} \cdot \exists \cdot \mathrm{BP} \cdot \exists \cdot \mathrm{P}$. Prove that these two classes are equal.