6.890: Fun with Hardness Proofs

Guest Lectures on PPAD

November 2014

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Menu

↳ Existence Theorems: Nash, Brouwer, Sperner

Menu

→ Existence Theorems: Nash, Brouwer, Sperner

Games and Equilibria

	1/2	1/2
Kick / Dive	Left	Right
1/2 Left	1 , -1	-1 , 1
1/2 Right	-1 , 1	1, -1

Penalty Shot Game

Equilibrium:

A pair of randomized strategies so that no player has incentive to deviate if the other stays put.

[von Neumann ’28]: It always exists in two-player zero-sum games.

$\Leftarrow$ Strong LP duality

\+ equilibrium can be computed in poly-time with Linear Programming

Games and Equilibria

		2/5	3/5
	Kick / Dive	Left	Right
1/2	Left	2 , -1	-1 , 1
1/2	Right	-1 , 1	1, -1

Equilibrium:

A pair of randomized strategies so that no player has incentive to deviate if the other stays put.

[Nash '50]: *An equilibrium exists in every game.*

no proof using LP duality known
no poly-time algorithm known, despite intense effort

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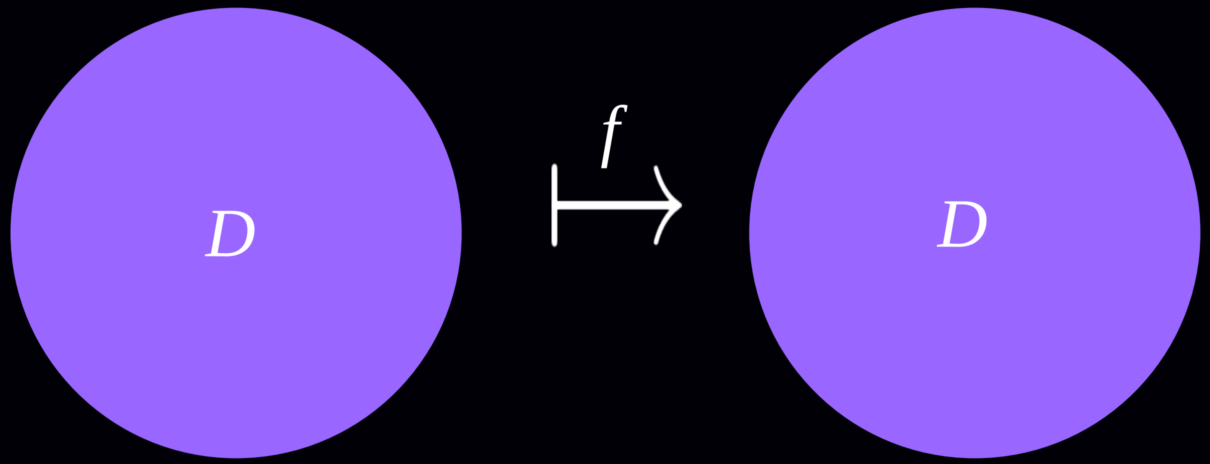
Brouwer's Fixed Point Theorem

[Brouwer 1910]: Let $f : D \longrightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

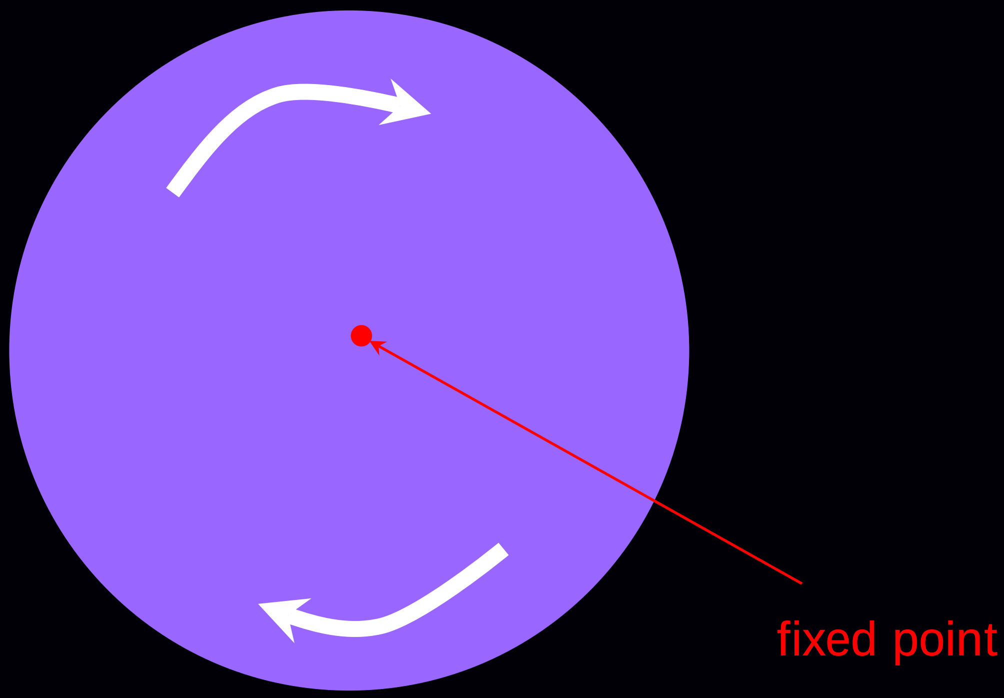
closed and bounded

Below we show a few examples, when D is the 2-dimensional disk.



N.B. All conditions in the statement of the theorem are necessary.

Brouwer's Fixed Point Theorem



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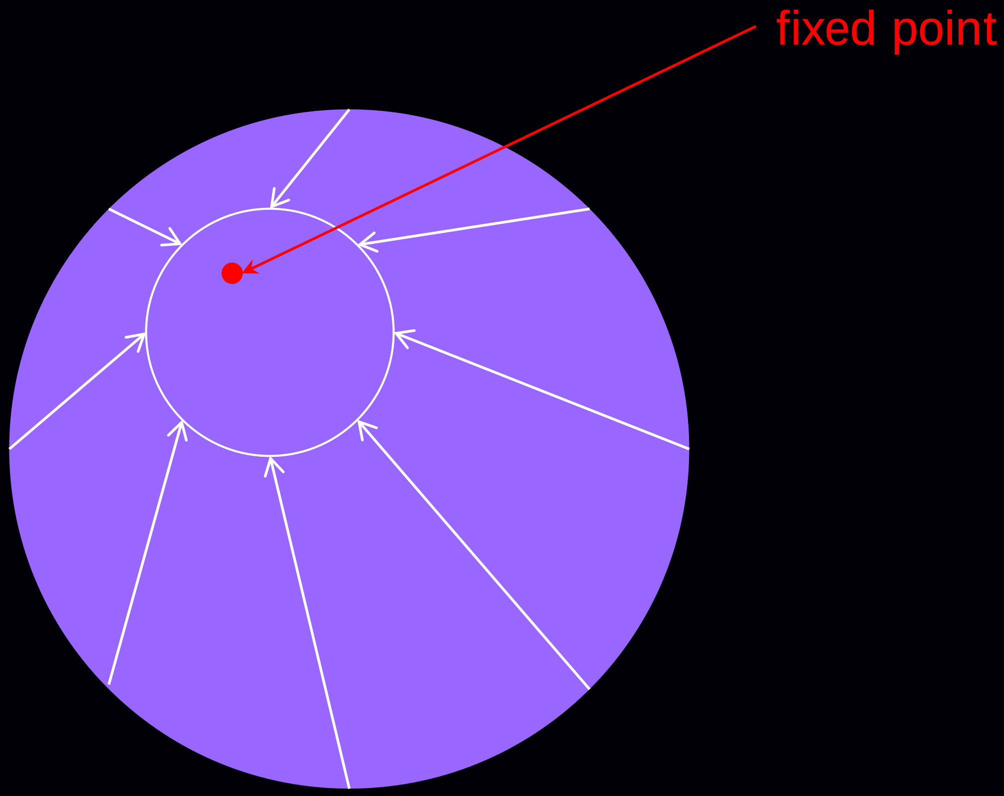

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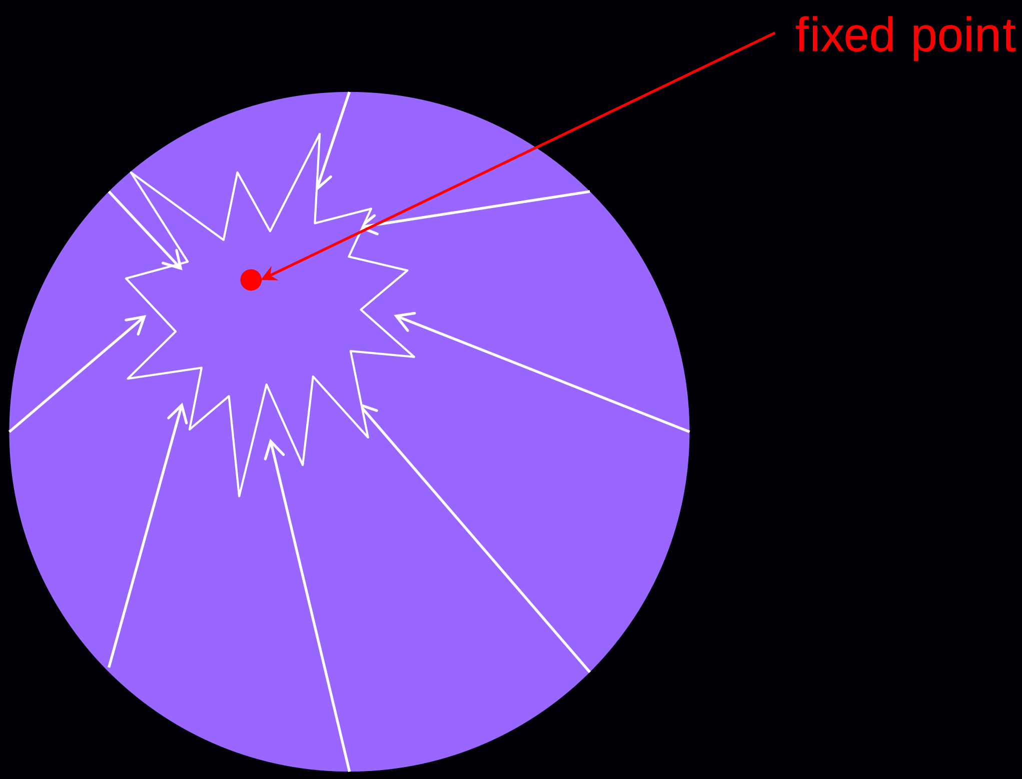


$Brouwer \Rightarrow Nash$

Visualizing Nash's Proof

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\mapsto$

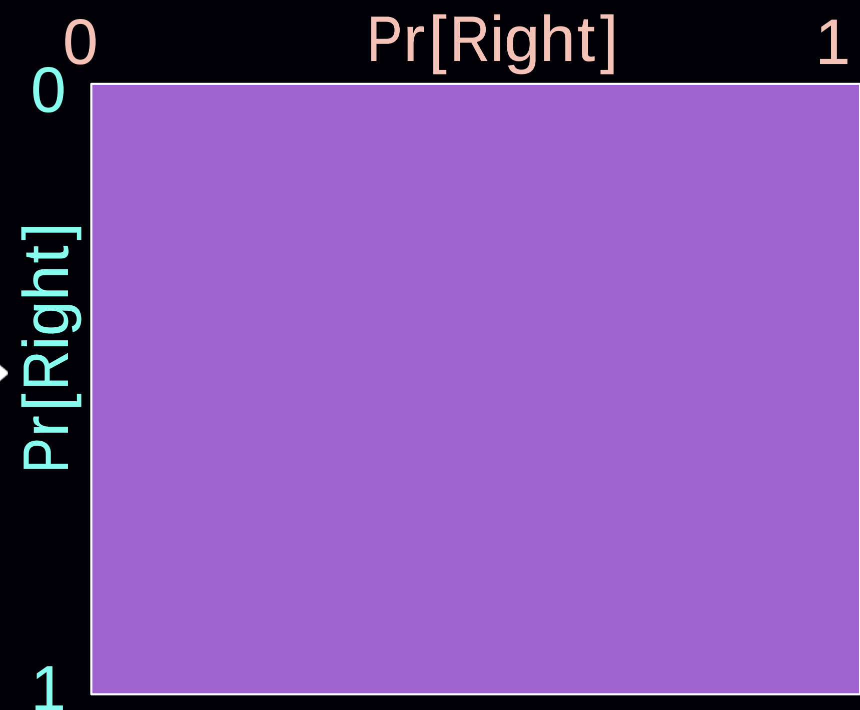
$f: [0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Visualizing Nash's Proof

Dive \ Kick	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$



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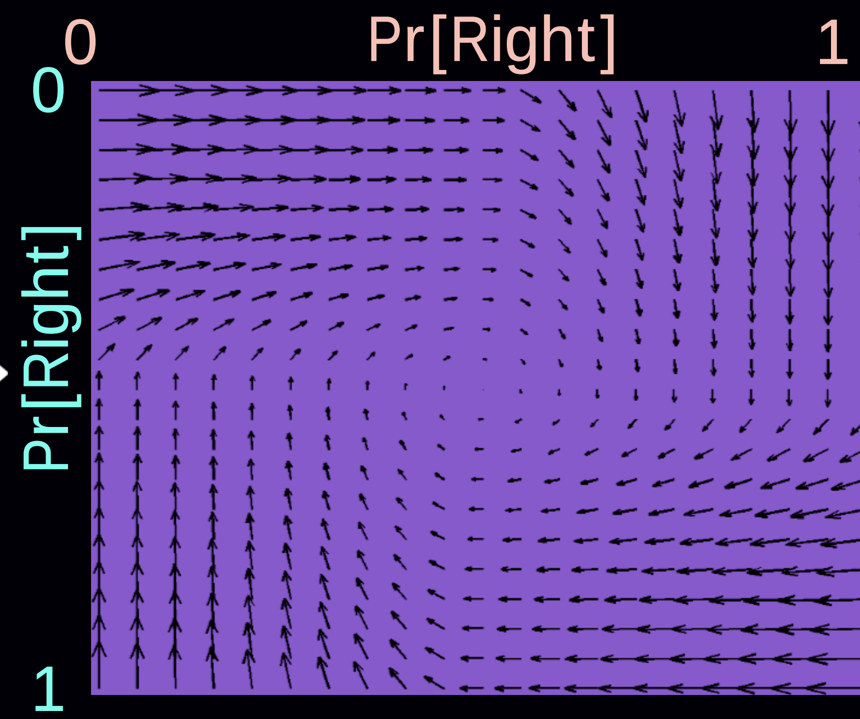

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Penalty Shot Game

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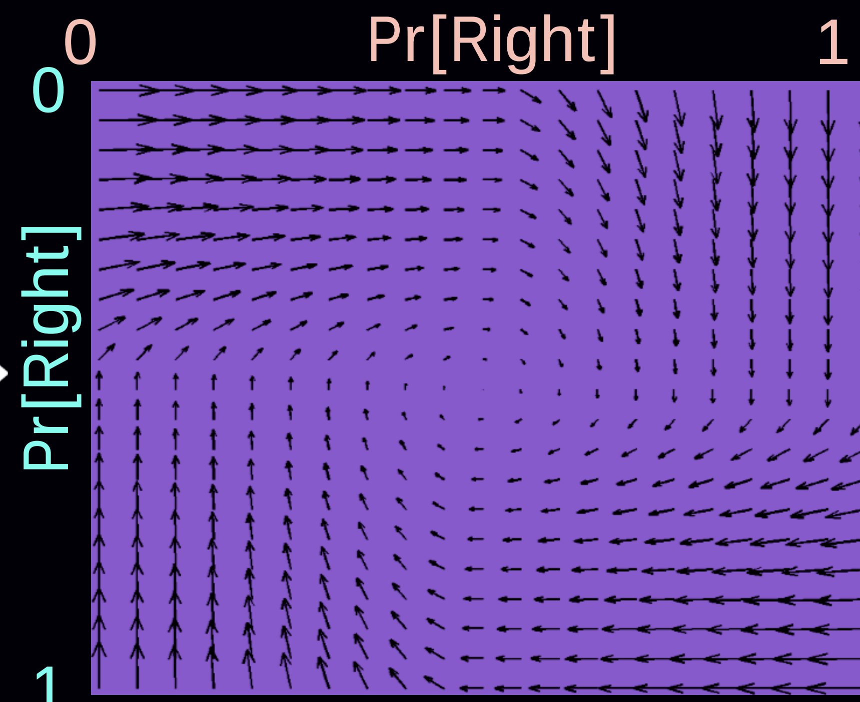
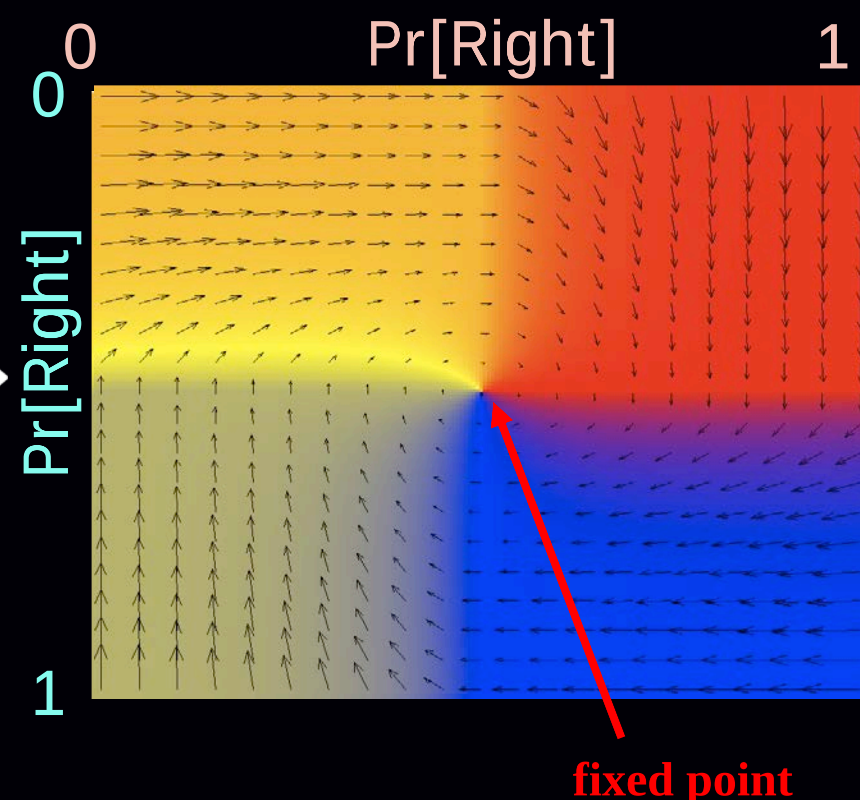


Visualizing Nash's Proof

		½	½
	Kick / Dive	Left	Right
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$



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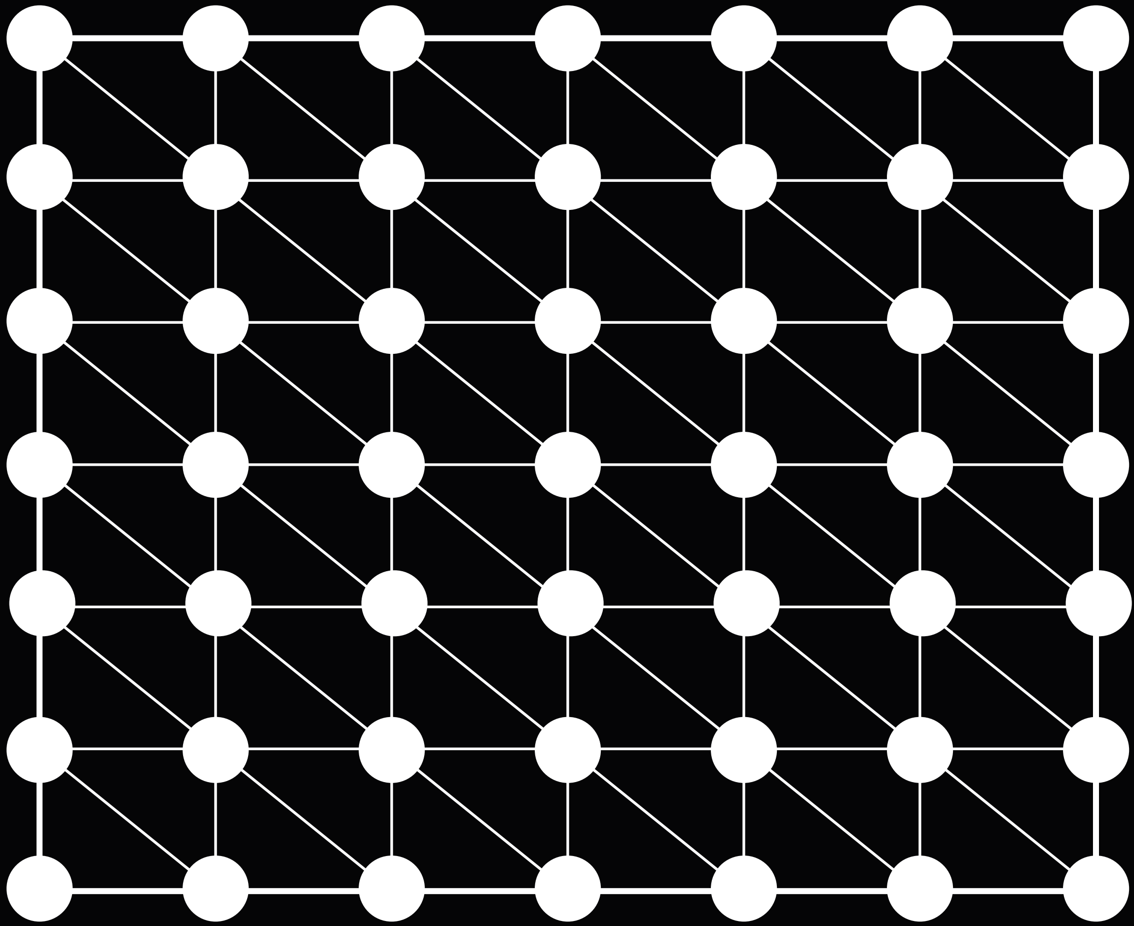
Sperner's Lemma (2-D)



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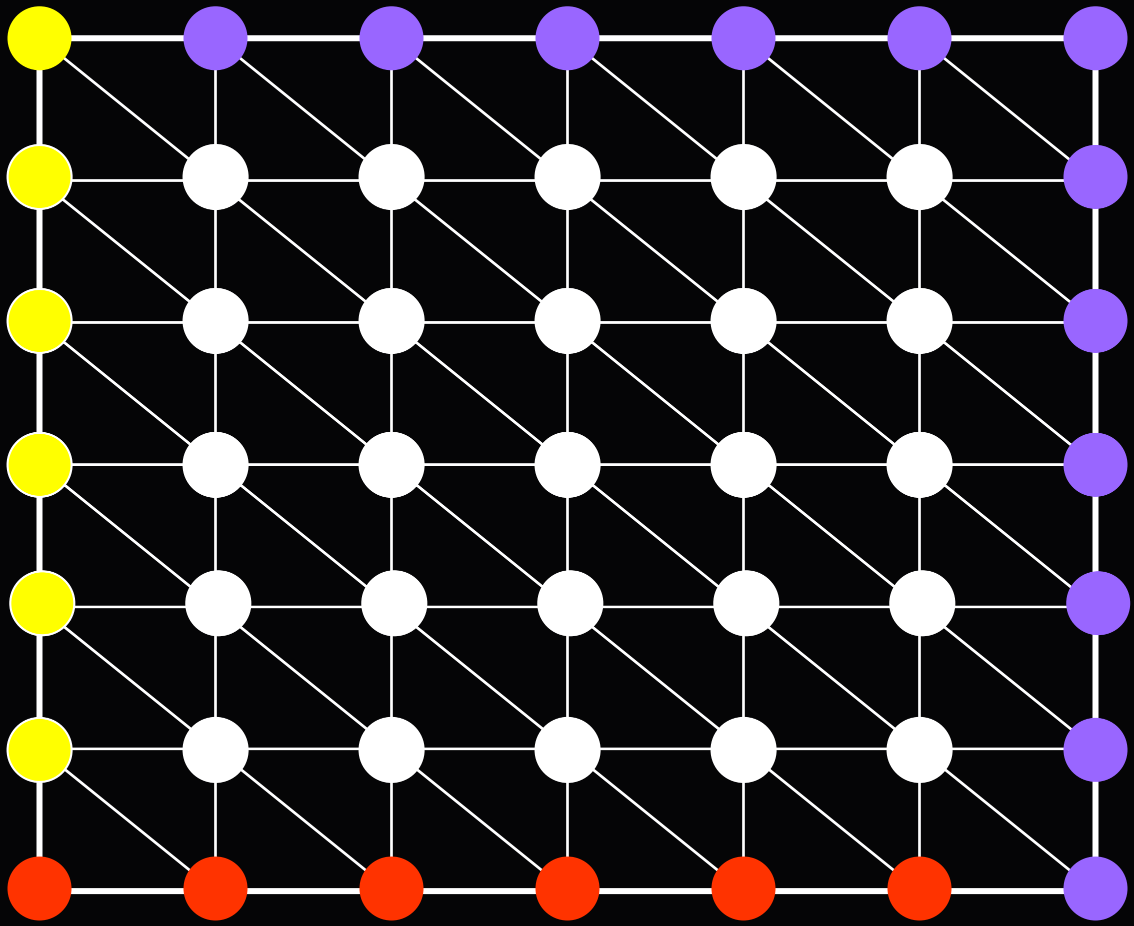

legal
boundary
coloring

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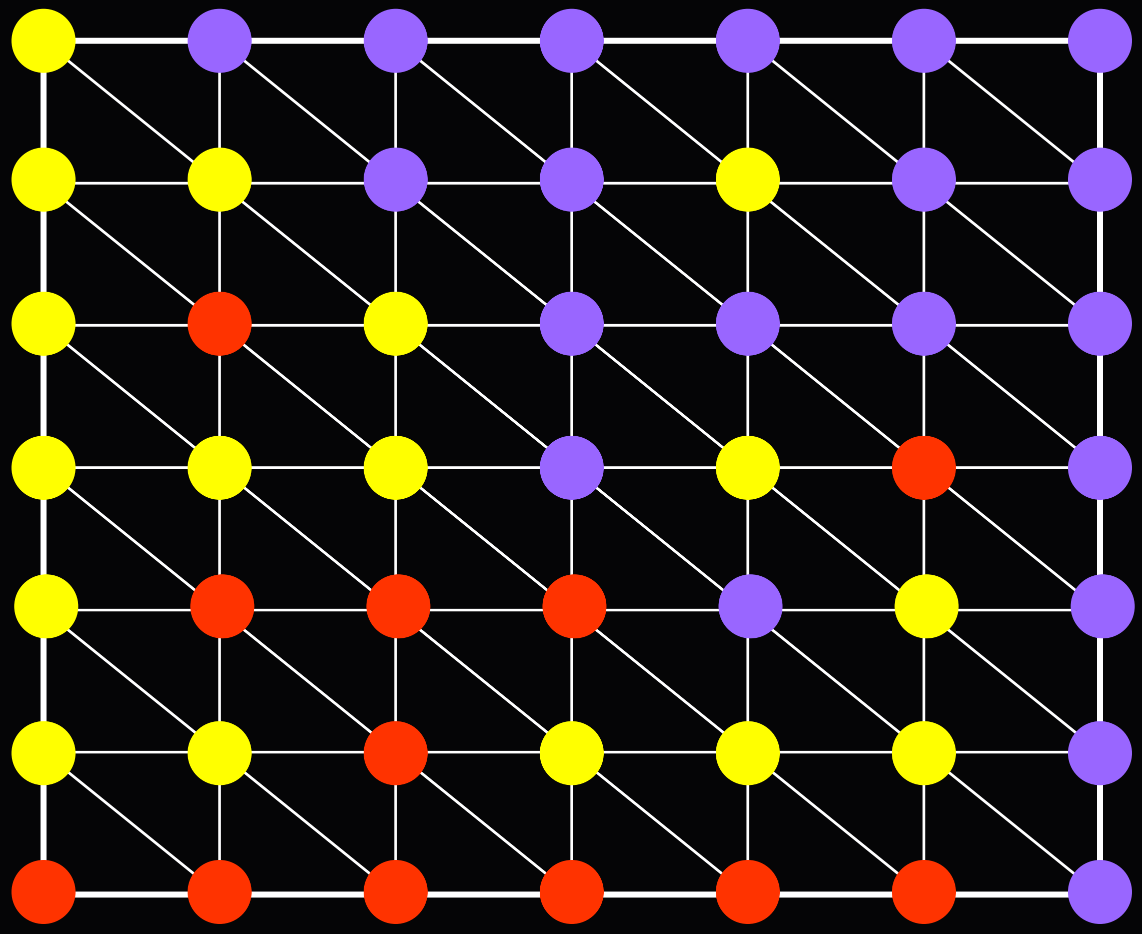

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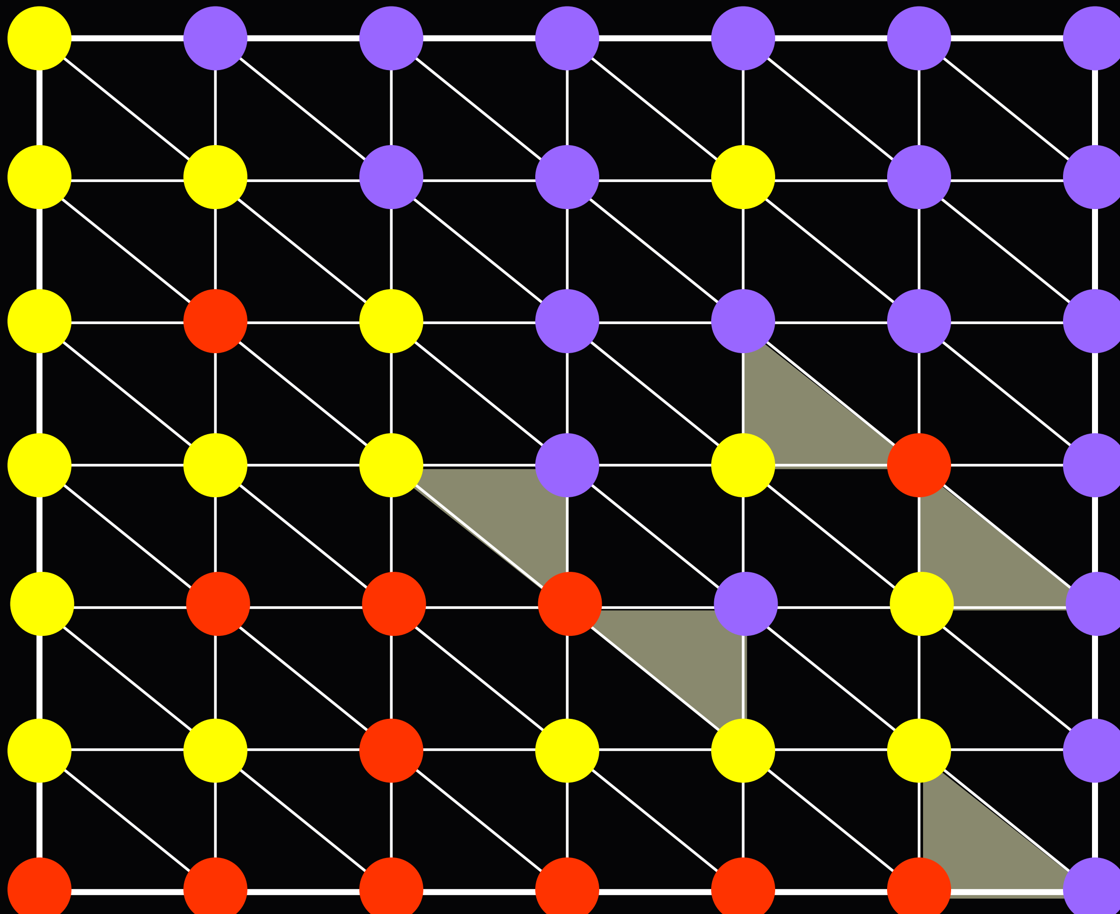
[Sperner 1928]: If the boundary is legally colored (and regardless how the internal nodes are colored), there exists a tri-chromatic triangle. In fact, an odd number of them.

Sperner's Lemma (2-D)



legal boundary coloring

[Sperner 1928]: If the boundary is legally colored (and regardless how the internal nodes are colored), there exists a tri-chromatic triangle. In fact, an odd number of them.

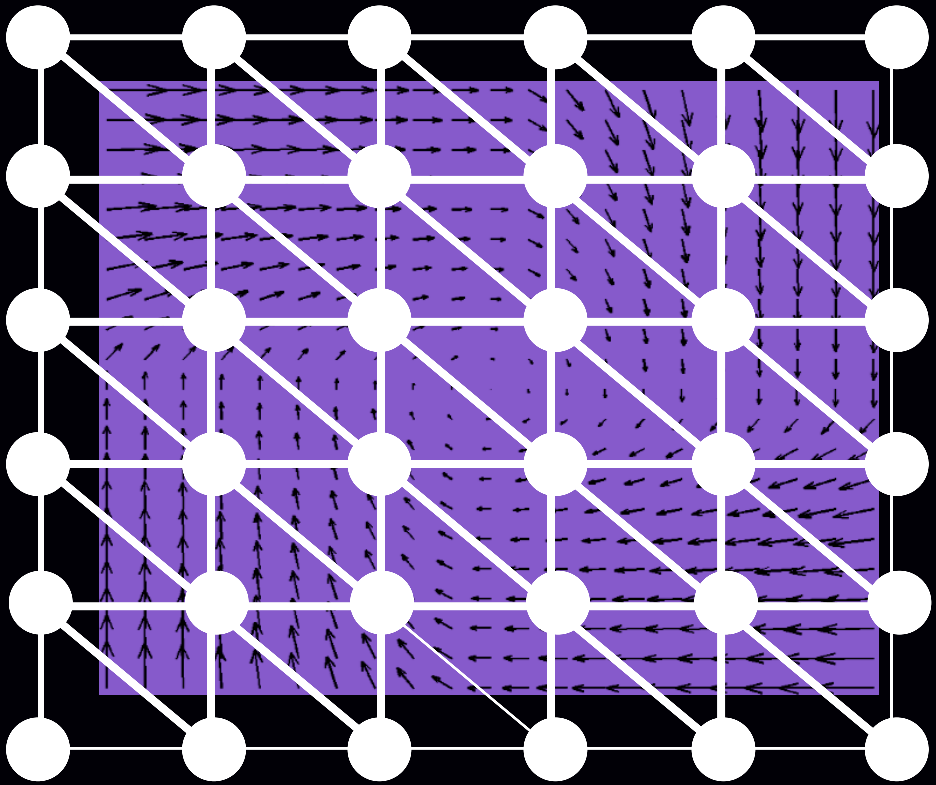
Sperner $\Rightarrow$ *Brouwer*

Sperner ⇒ Brouwer

Given $f : [0,1]^2 \rightarrow [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$,



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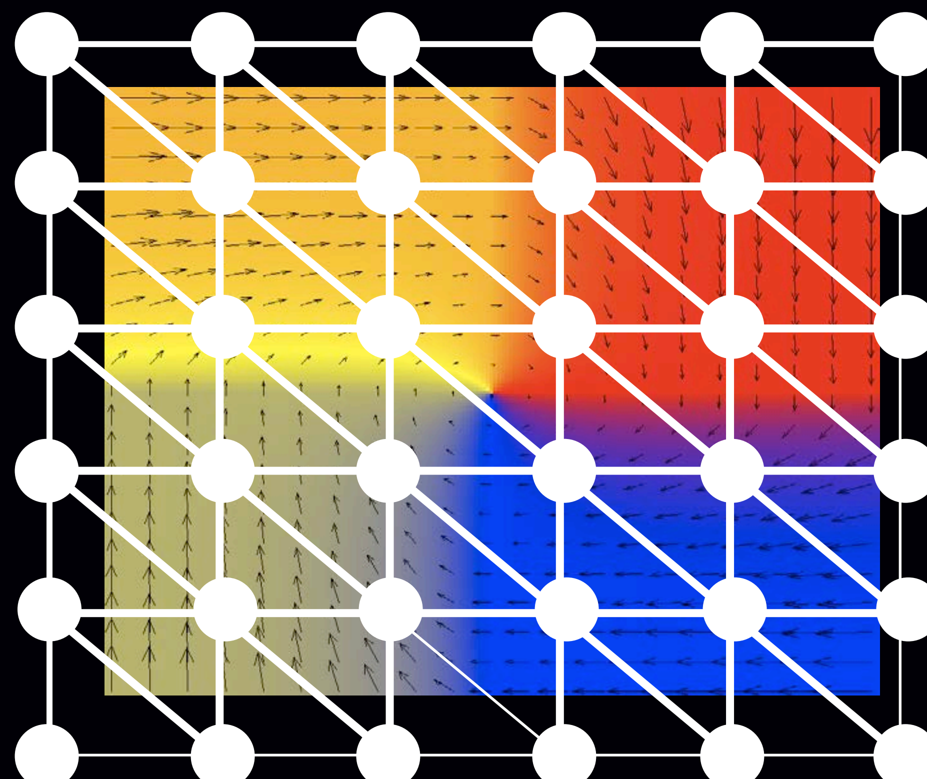
For 1: Triangulate $[0,1]^2$, then color points according to the direction of $f(x)-x$.



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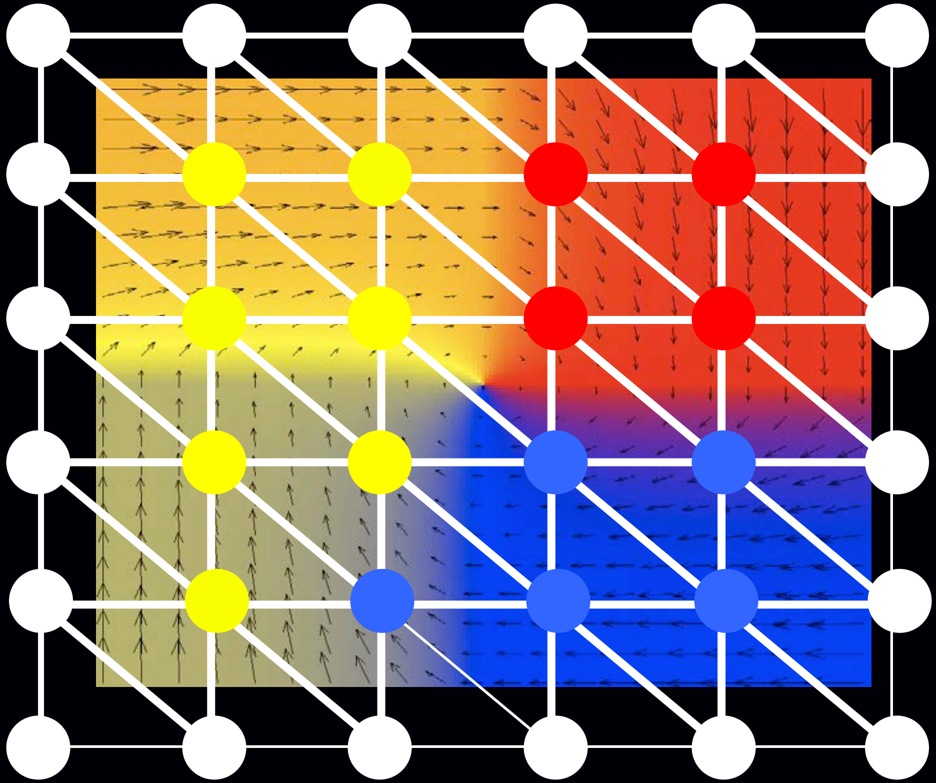

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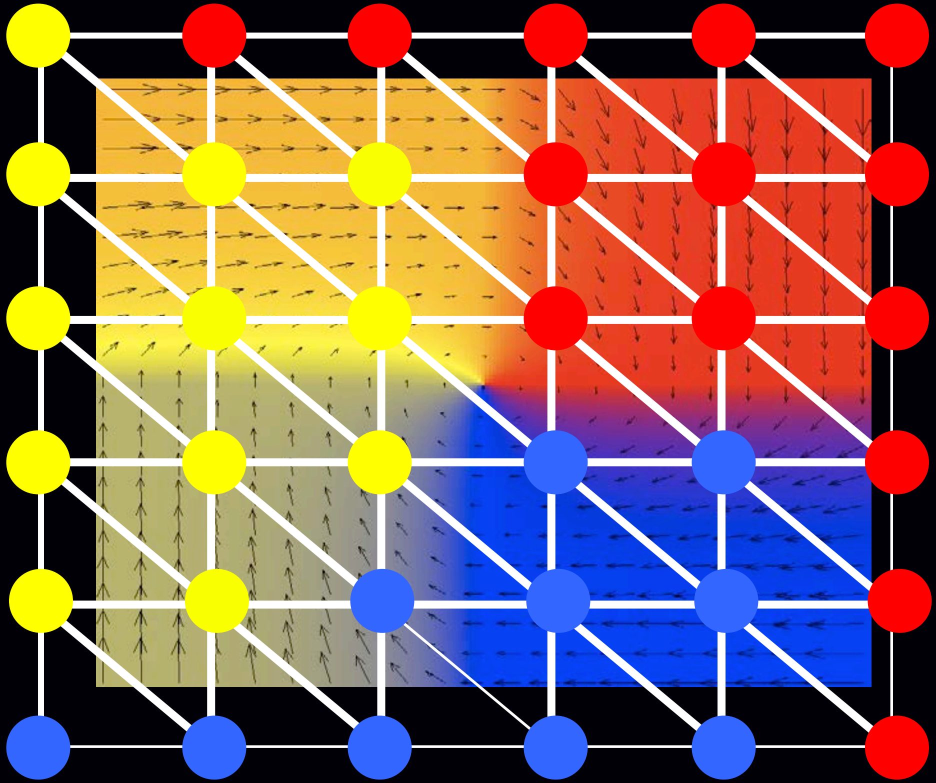


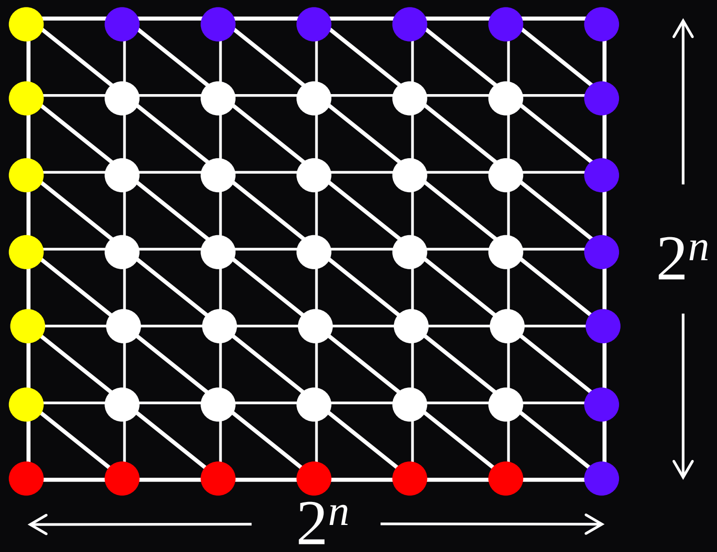
Menu

- Existence Theorems: Nash, Brouwer, Sperner
- Total Search Problems in NP

SPERNER

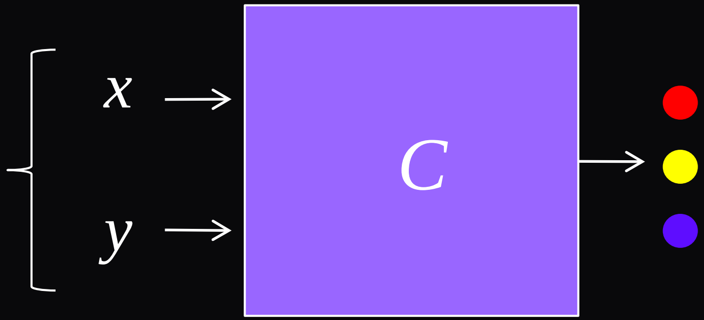
INPUT:

(i) Grid of side 2^n:



(ii) Suppose boundary has standard coloring, and colors of internal vertices are given by a circuit:

input: the coordinates of a point *(n bits each)*



OUTPUT: A tri-chromatic triangle.

NASH

INPUT: (i) A Game defined by

- the number of players n;
- an enumeration of the strategy set S_p of every player $p = 1,\ldots, n$;
- the utility function $u_p : S \longrightarrow \mathbb{R}$ of every player.

(ii) An approximation requirement ε

OUTPUT: An ε-Nash equilibrium of the game.

i.e. the expected payoff of every player is within additive ε from the optimal expected payoff given the others' strategies

* Approximation: Already in 1951, Nash provides a 3-player game whose unique equilibrium is irrational. This motivates our definition of the problem in terms of approximation.

** 2-player Games: 2-player games always have a rational equilibrium of polynomial description complexity in the size of the game. So we can also define the exact NASH problem for 2-player games.

Function NP (FNP)

A *search problem* L is defined by a relation $R_L \subseteq \{0,1\}^* \times \{0,1\}^*$ such that $(x, y) \in R_L$ iff y is a solution to x

A search problem is called *total* iff $\forall x.\ \exists\ y$ such that $(x, y) \in R_L$.

A search problem $L \in$ FNP iff there exists a poly-time algorithm $A_L(\cdot, \cdot)$ and a polynomial function $p_L(\cdot)$ such that

(i) $\forall x, y$: $A_L(x, y)=1 \Leftrightarrow (x, y) \in R_L$

(ii) $\forall x$: $\exists y$ s.t. $(x, y) \in R_L \Rightarrow \exists z$ with $|z| \leq p_L(|x|)$ s.t. $(x, z) \in R_L$

TFNP = $\{L \in \text{FNP} \mid L \text{ is total}\}$

SPERNER, NASH, BROUWER $\in$ FNP.

FNP-completeness

A search problem $L \in$ FNP, associated with A_L and p_L, is ***poly-time (Karp) reducible*** to another problem $L' \in$ FNP, associated with $A_{L'}$ and $p_{L'}$, iff there exist efficiently computable functions *f*, *g* such that

(i) $f : \{0,1\}^* \rightarrow \{0,1\}^*$ maps inputs x to L into inputs $f(x)$ to L'

(ii)

$$\forall\, x,y: A_{L'}(f(x), y)=1 \quad \Rightarrow \quad A_L(x, g(y))=1$$

$$\forall\, x: A_{L'}(f(x), y)=0, \forall\, y \Rightarrow \quad A_L(x, y)=0, \forall\, y$$

can't reduce SAT to SPERNER, NASH or BROUWER

A search problem L is ***FNP-complete*** iff

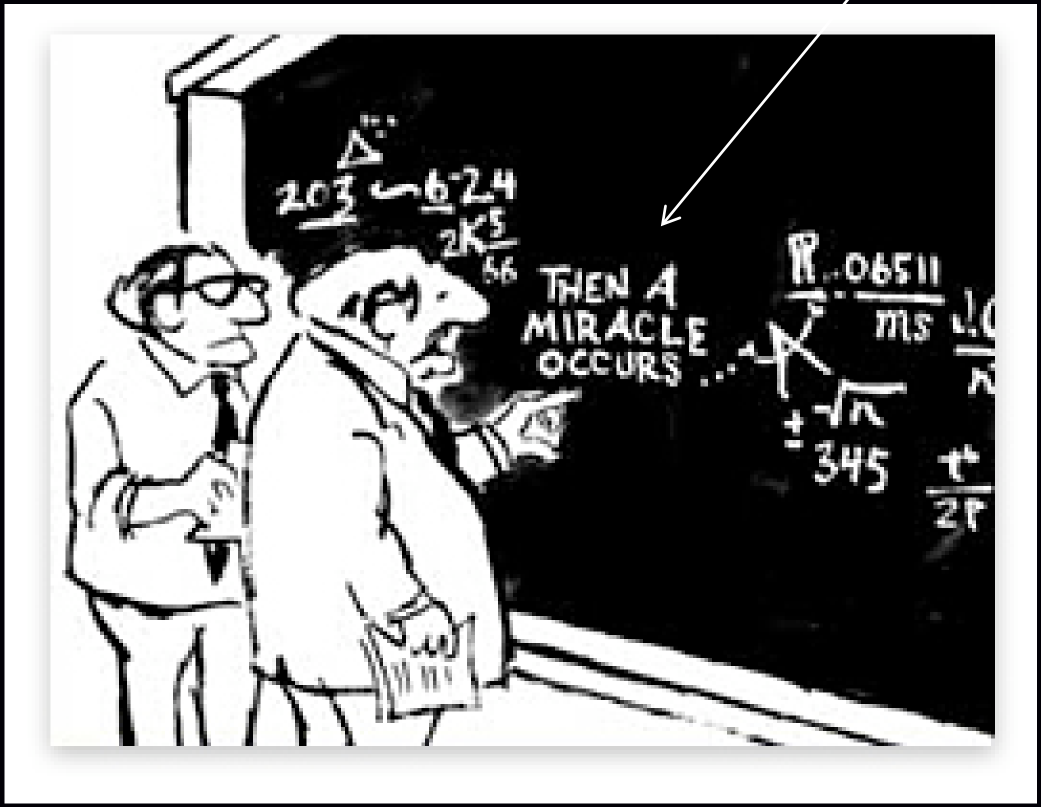
e.g. SAT

$L \in$ FNP

L' is poly-time reducible to L, for all $L' \in$ FNP

A Complexity Theory of Total Search Problems ?

??



A Complexity Theory of Total Search Problems ?

100-feet overview of our methodology:

1. identify the combinatorial argument of existence, responsible for making these problems total;

2. define a complexity class inspired by the argument of existence;

3. make sure that the complexity of the problem was captured as tightly as possible (via completeness results).

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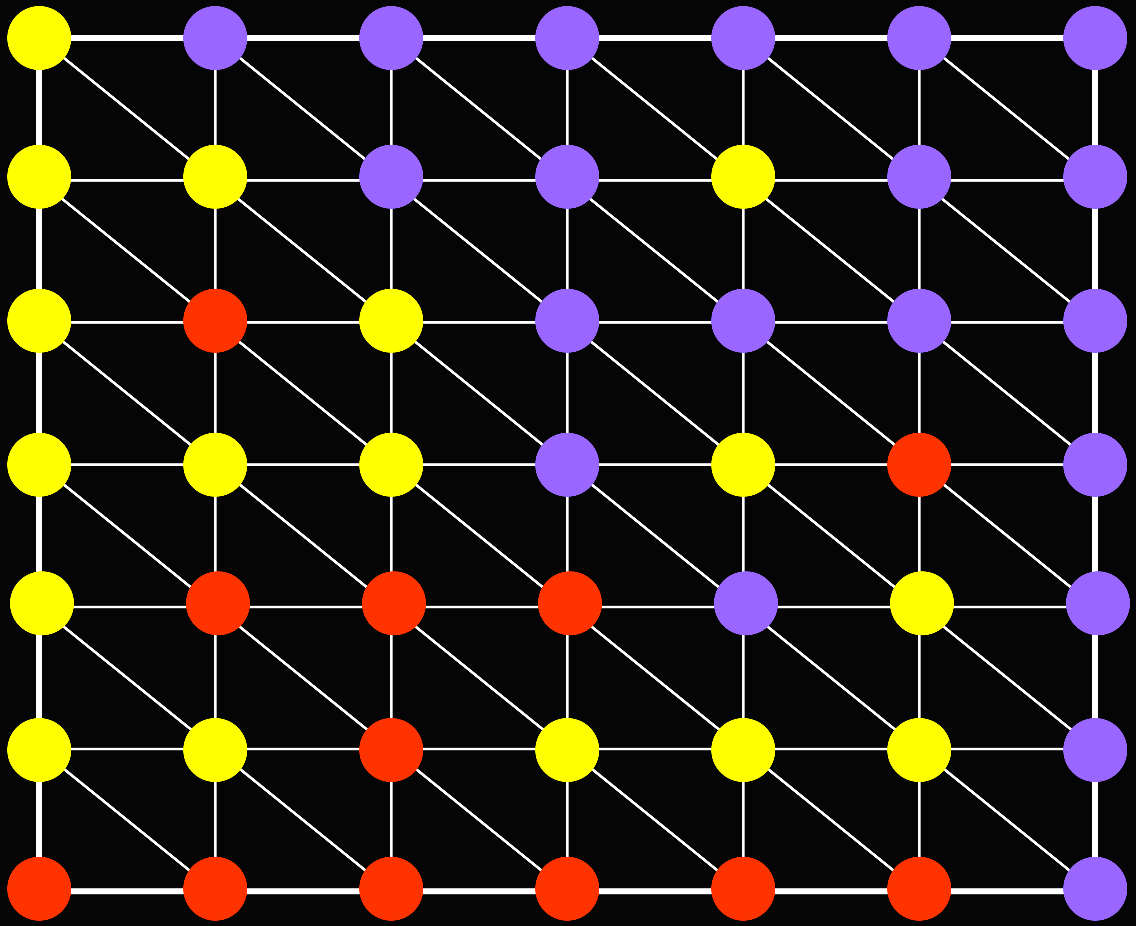
- Total Search Problems in NP
- Identifying the Combinatorial Core

Proof of Sperner's Lemma

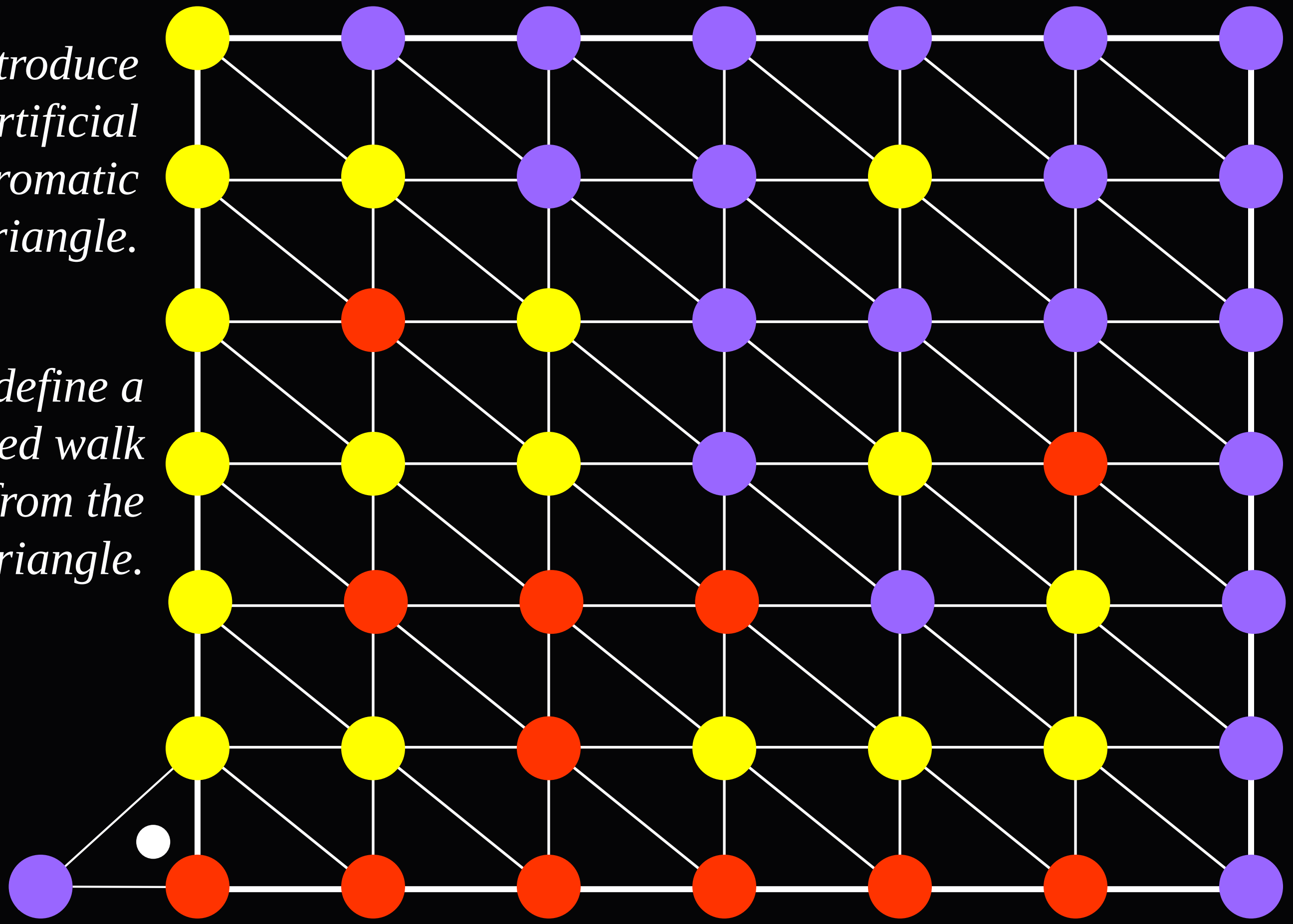


[Sperner 1928]: If the boundary is legally colored (and regardless how the internal nodes are colored), there exists a tri-chromatic triangle. In fact, an odd number of them.

Proof of Sperner's Lemma

1. We introduce an artificial tri-chromatic triangle.

2. We define a directed walk starting from the artificial triangle.



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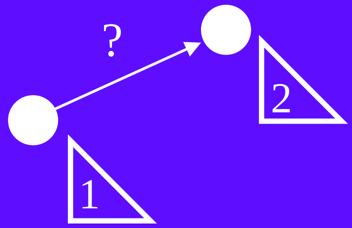
Proof of Sperner's Lemma

Set of Triangles

Transition Rule:

*If ∃ **red** - **yellow** door cross it keeping **yellow** on your left hand.*



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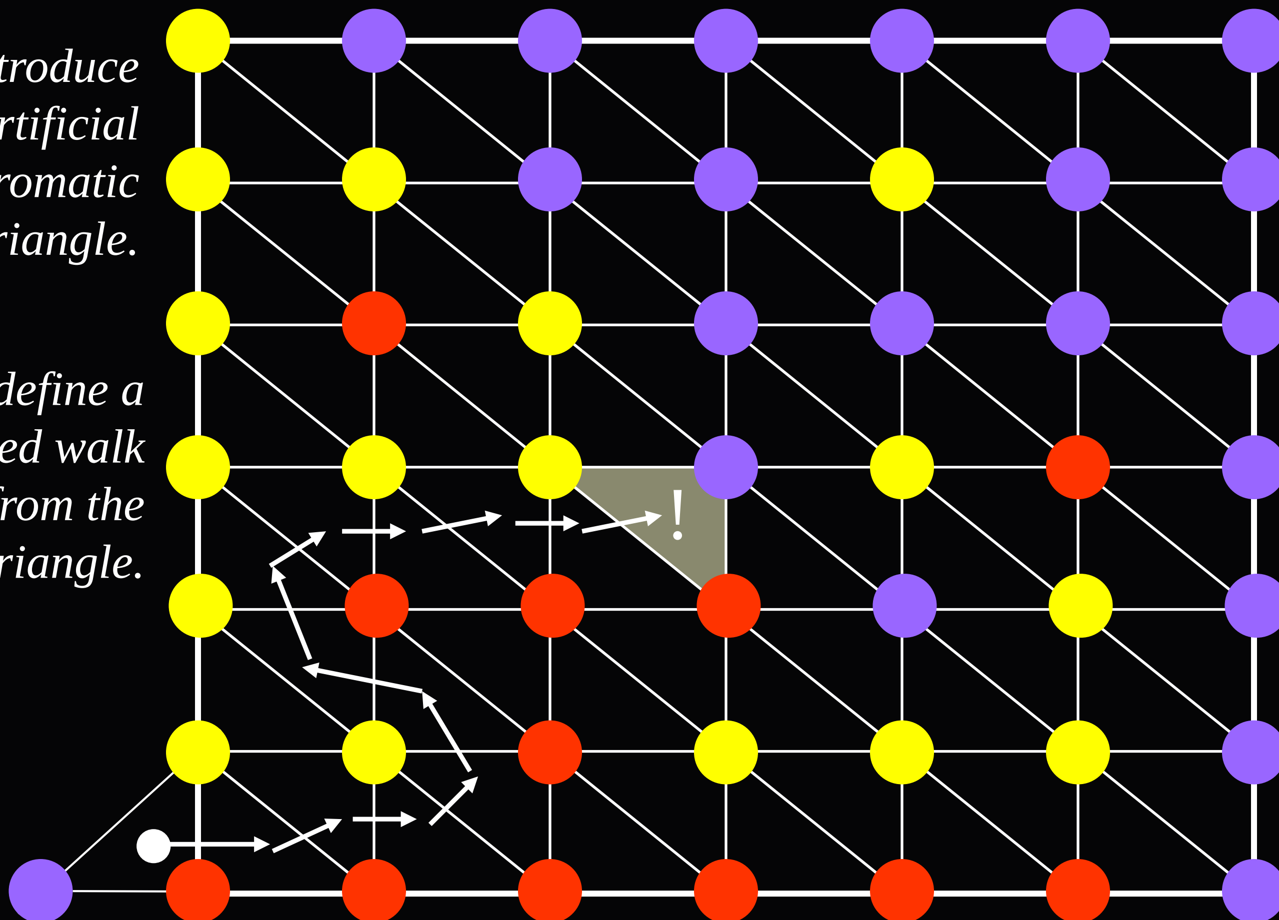

Claim: *The walk cannot exit the square, nor can it loop into itself.*

Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…

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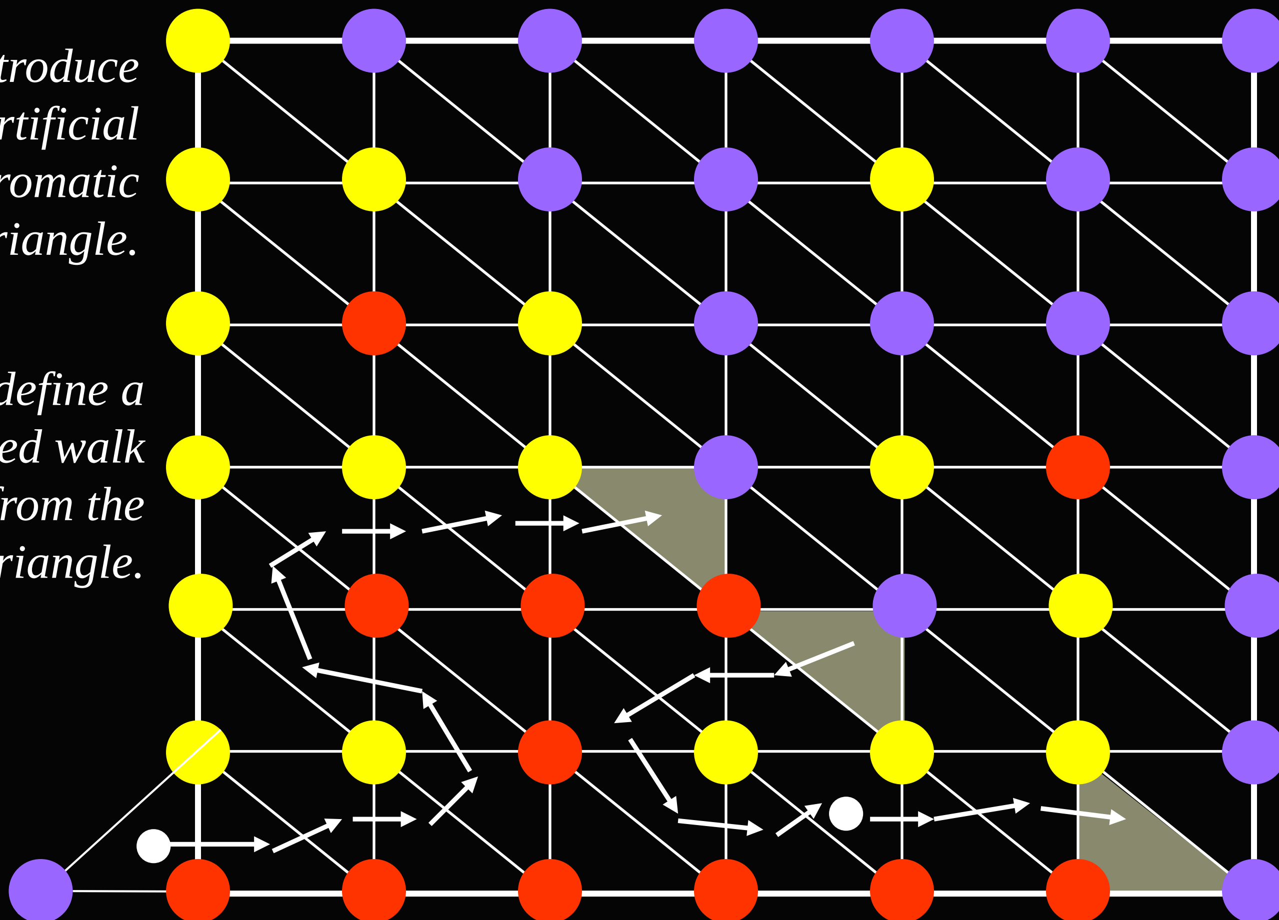

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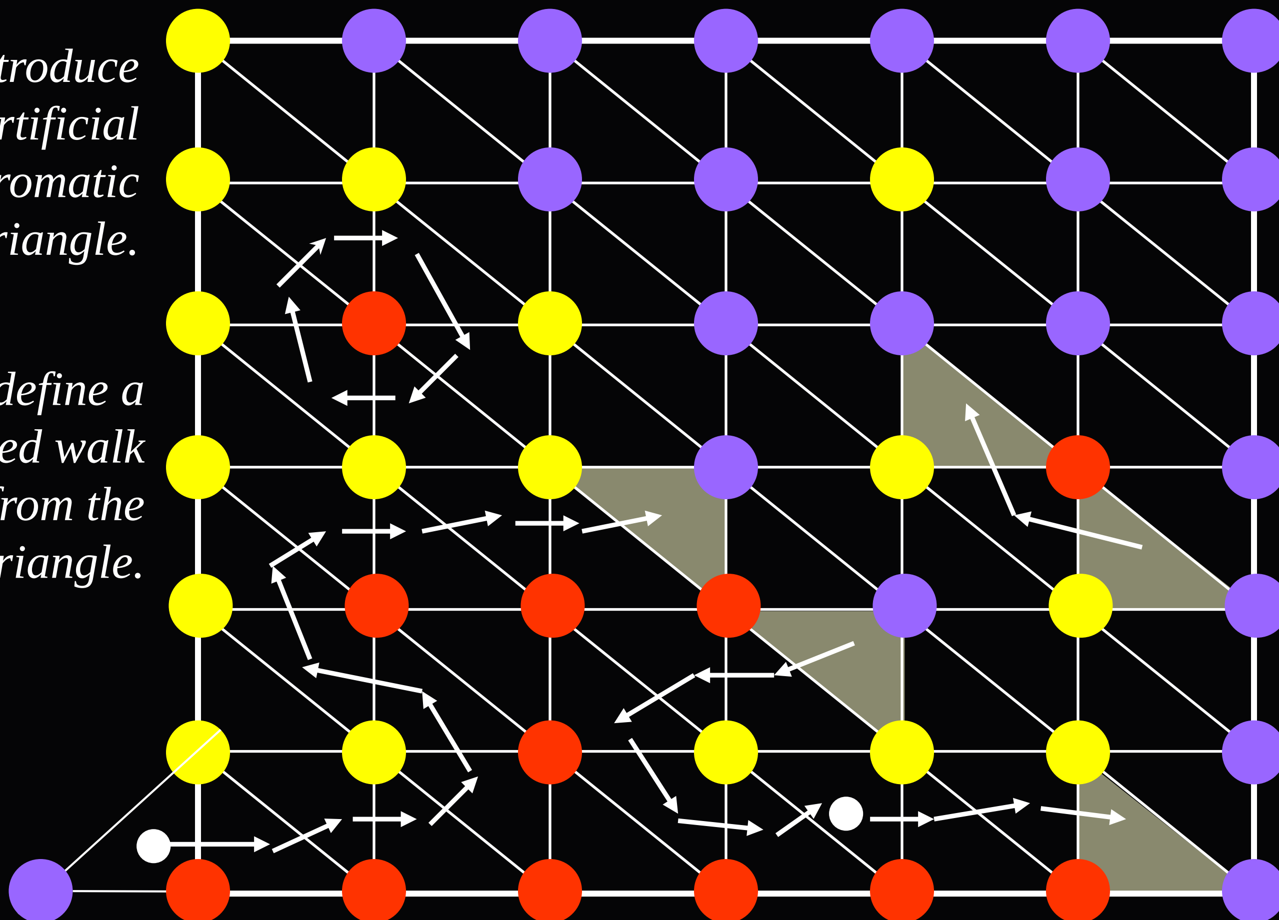

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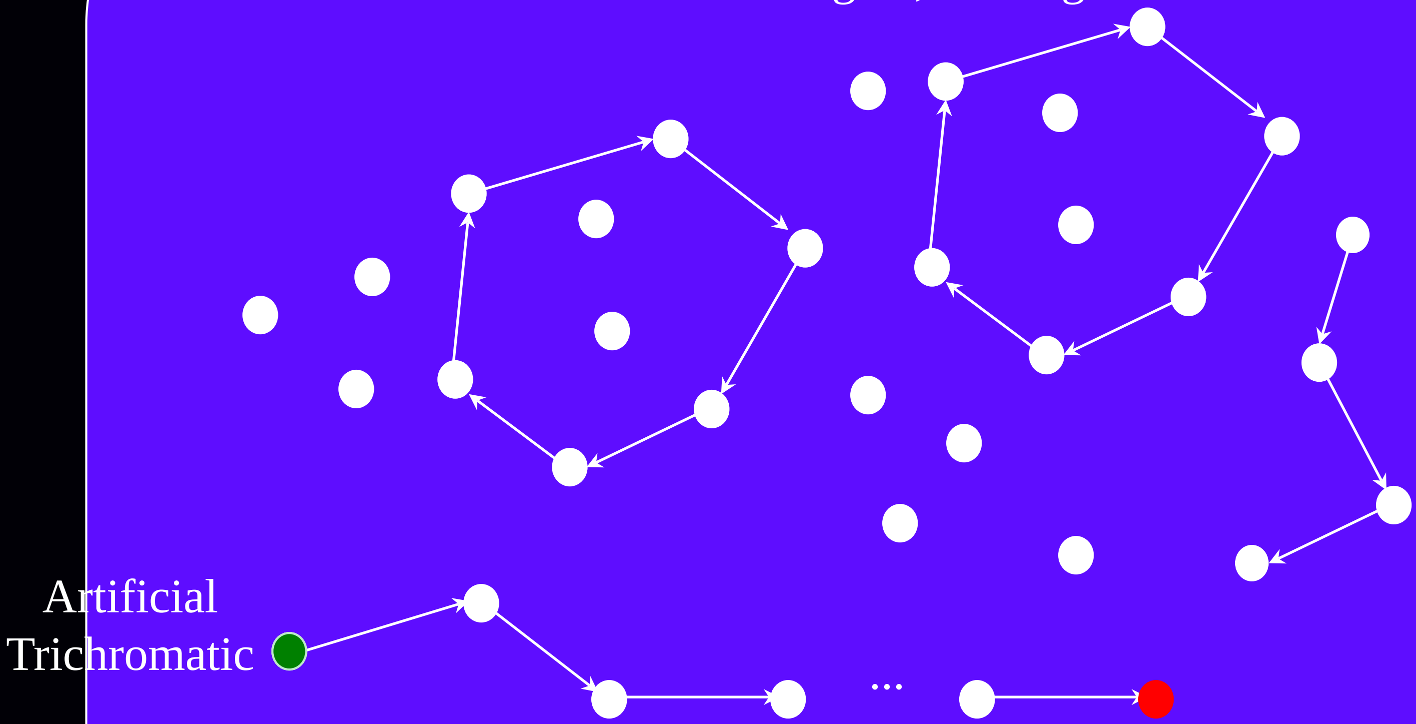
A directed parity argument

Vertices of Graph ≡ Triangles
all vertices have in-degree, out-degree ≤ 1



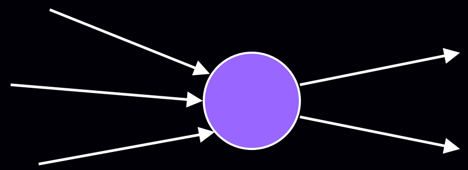
degree 1 vertices: trichromatic triangles
degree 2 vertices: no blue, non-trichromatic
degree 0 vertices: all other triangles

Proof: ∃ at least one trichromatic (artificial one) ➔ ∃ another trichromatic

The Non-Constructive Step

An easy parity lemma:

A directed graph with an unbalanced node (a node with indegree ≠ outdegree) must have another.



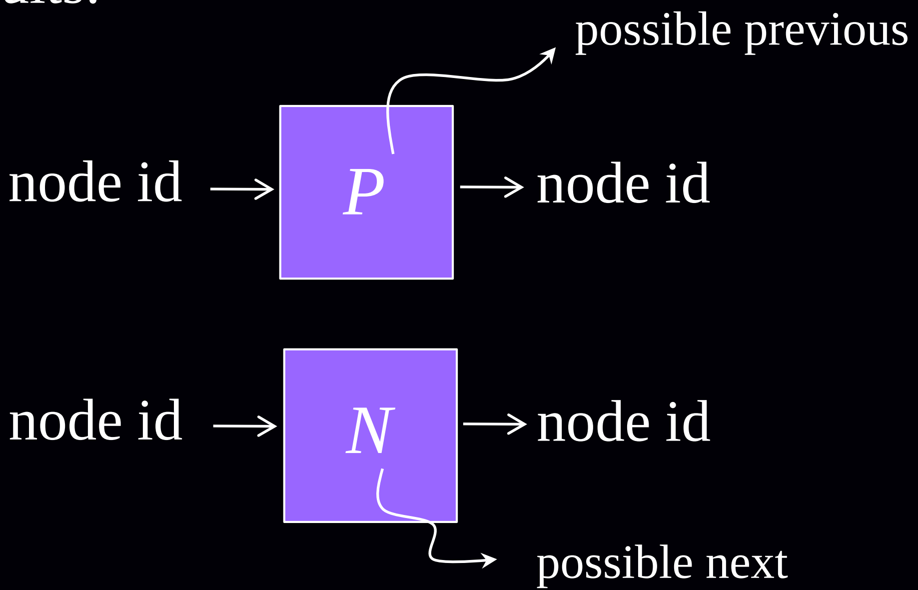
But, wait, why is this non-constructive?

Given a directed graph and an unbalanced node, isn't it trivial to find another unbalanced node?

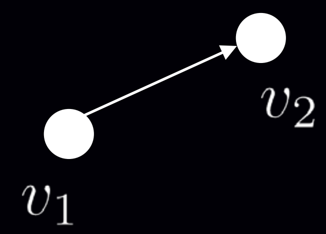
The graph can be exponentially large, but has succinct description…

The PPAD Class [Papadimitriou '94]

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:



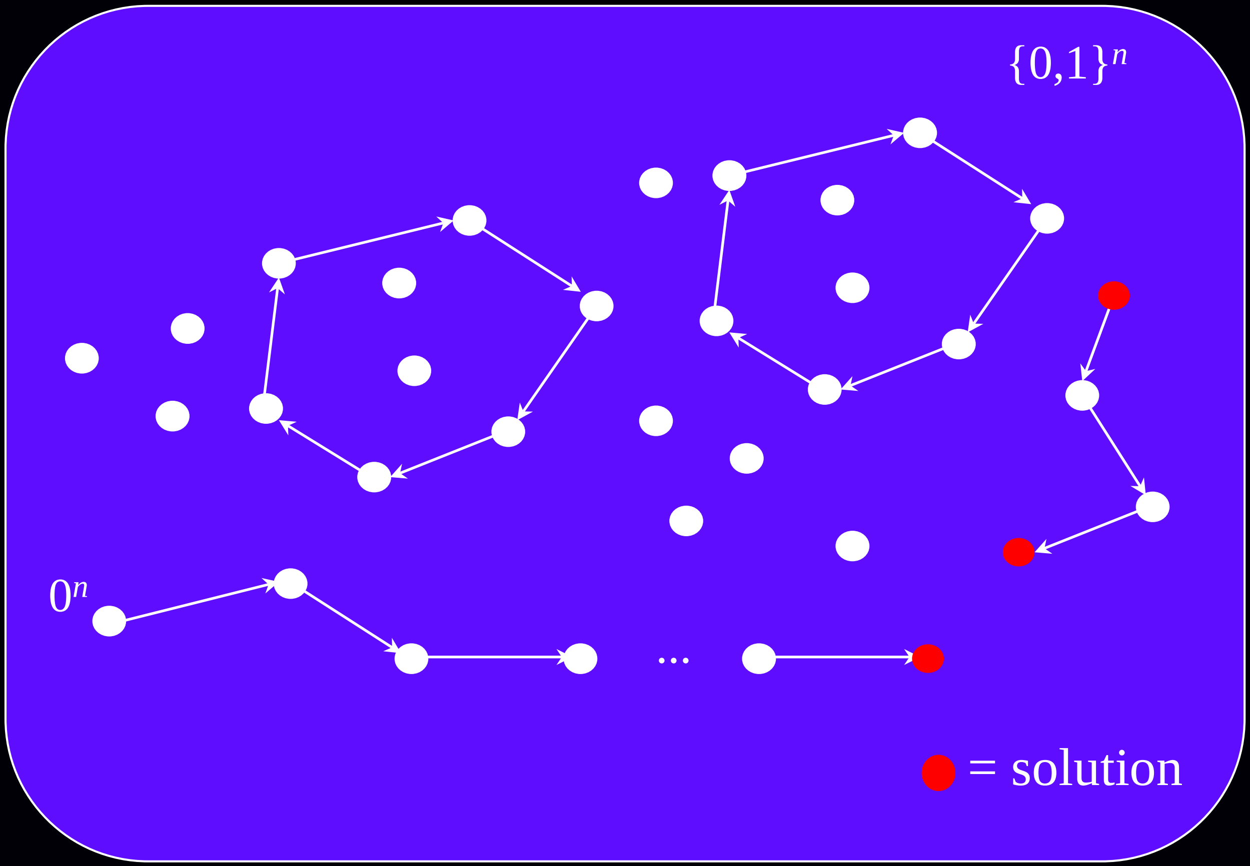
$$P(v_2) = v_1 \wedge N(v_1) = v_2$$

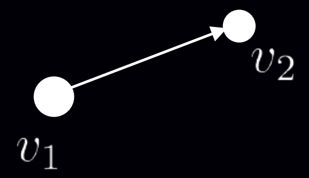


END OF THE LINE: Given P and N: If 0^n is an unbalanced node, find another unbalanced node. Otherwise output 0^n.

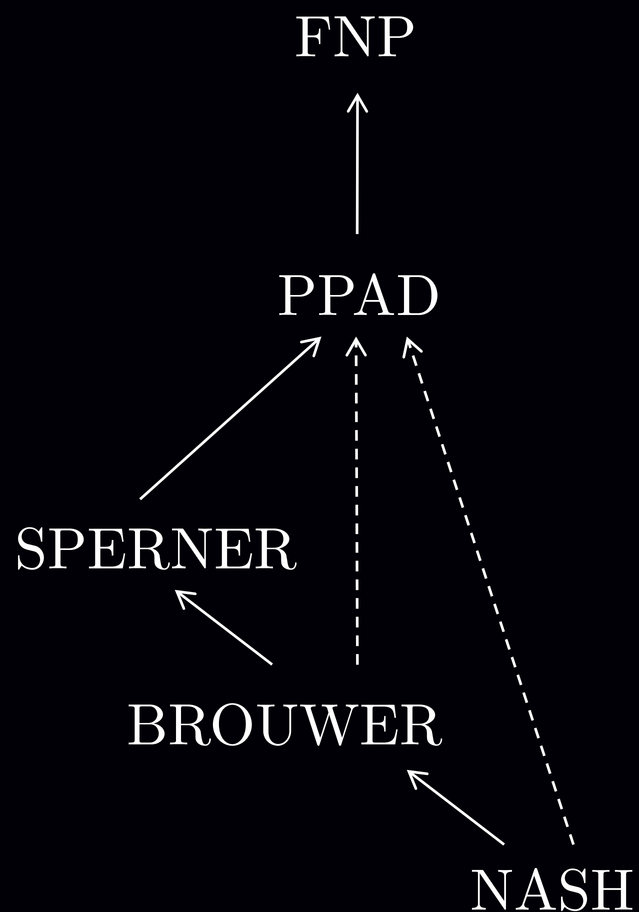
PPAD = { *Search problems in* FNP *reducible to* END OF THE LINE }

END OF THE LINE





$\Leftrightarrow \quad P(v_2) = v_1 \wedge N(v_1) = v_2$


FNP
PPAD
SPERNER
BROUWER
NASH

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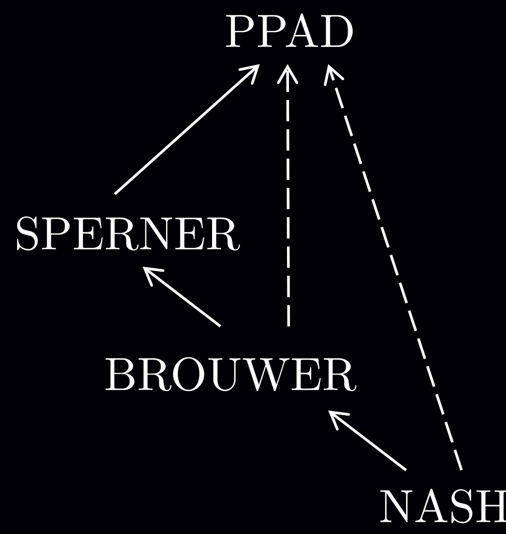
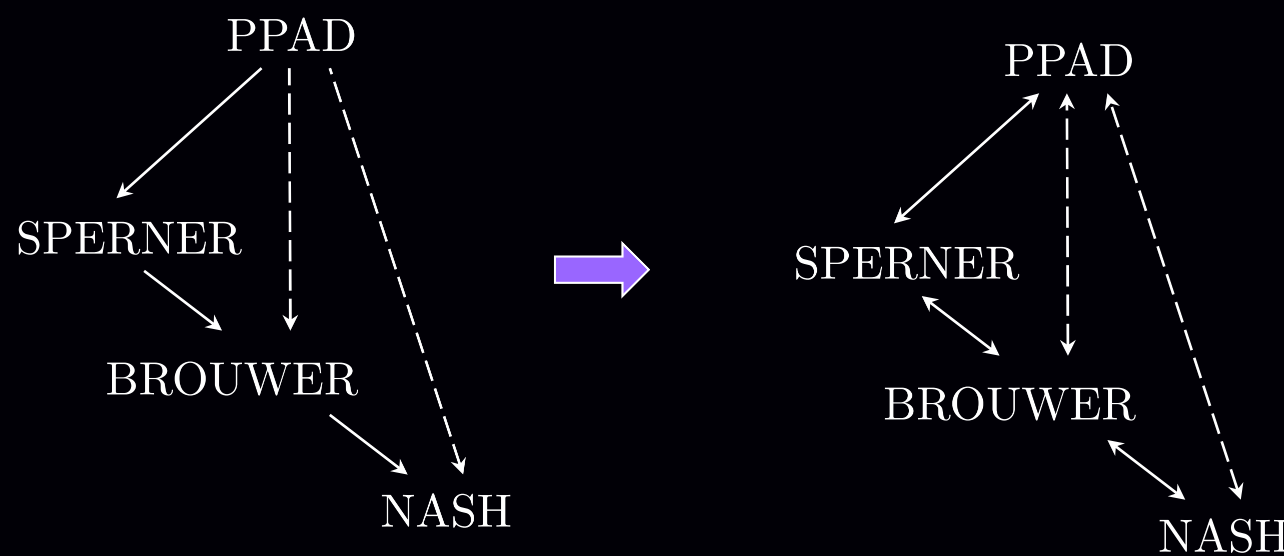
- Identifying the Combinatorial Core
- Litmus Test: PPAD-completeness Results

Inclusions that are easy to establish:

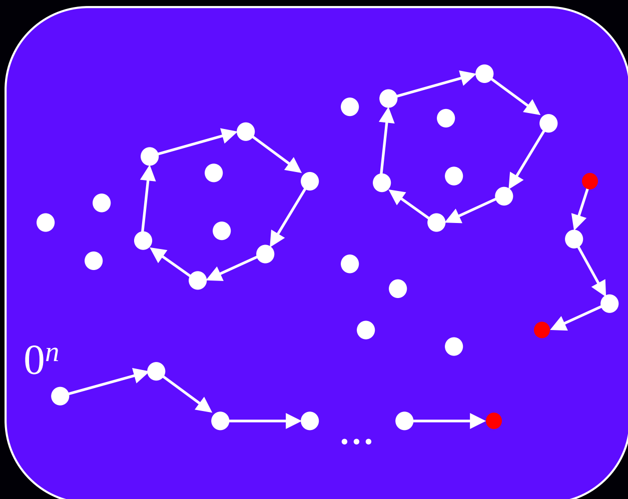


[Daskalakis-Goldberg-Papadimitriou’06]:

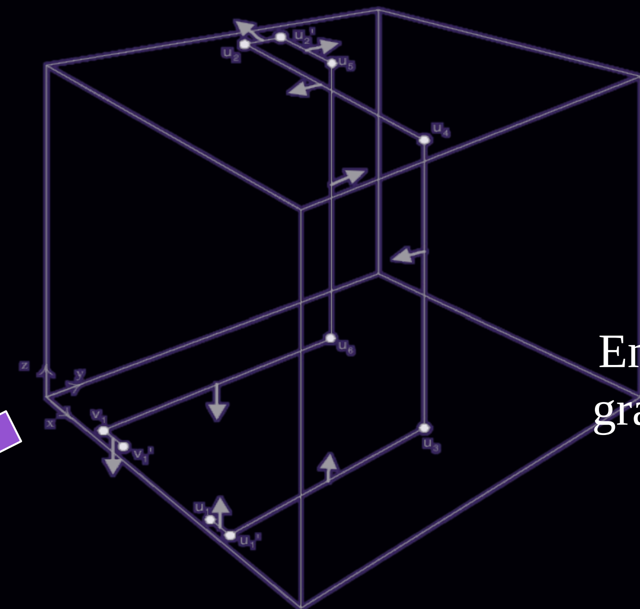


PPAD-Completeness of NASH

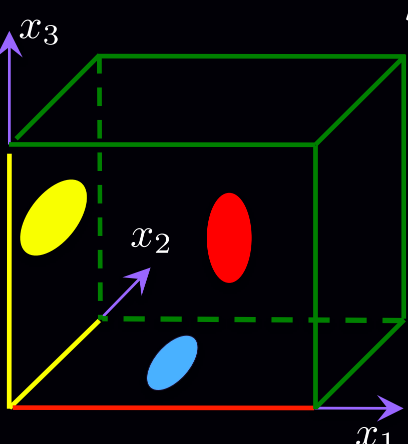
[Daskalakis, Goldberg, Papadimitriou'06]



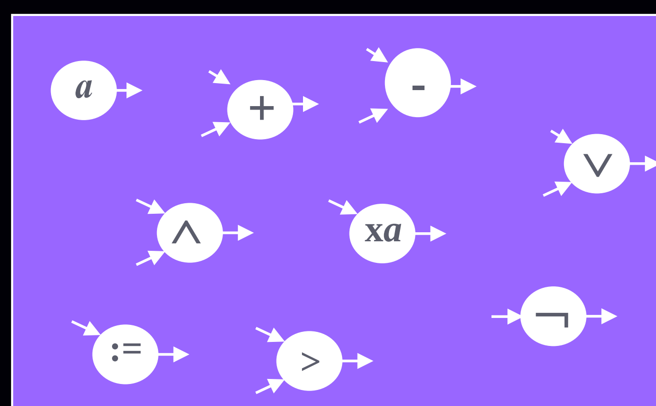
Generic PPAD



Embed PPAD graph in $[0,1]^3$



3D-SPERNER

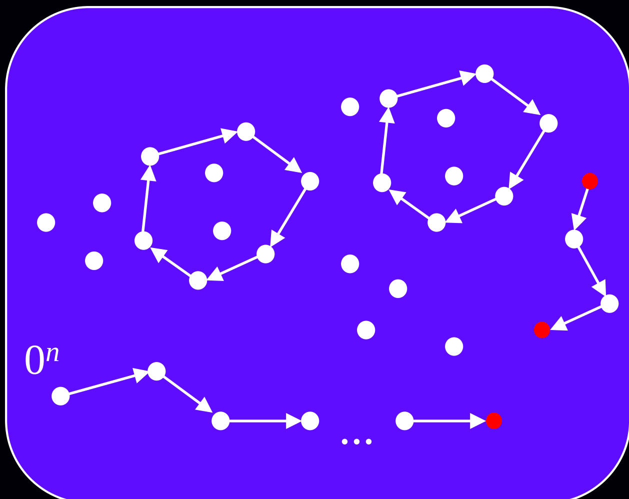


ArithmCircuitSAT

NASH

PPAD-Completeness of NASH

[Daskalakis, Goldberg, Papadimitriou'06]



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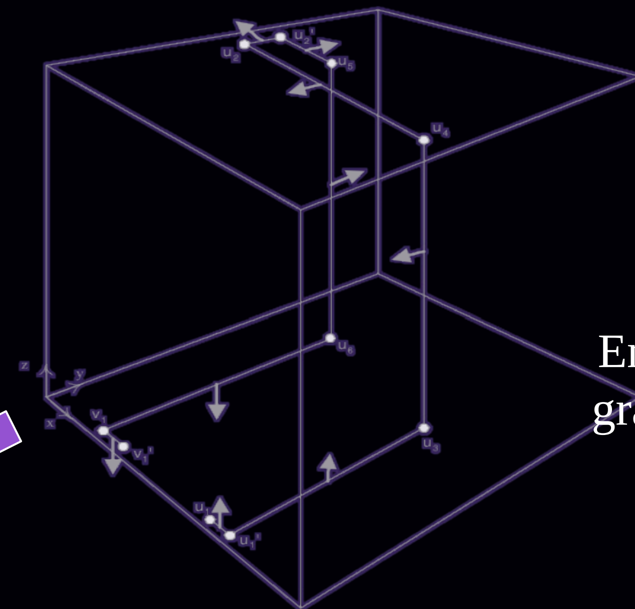

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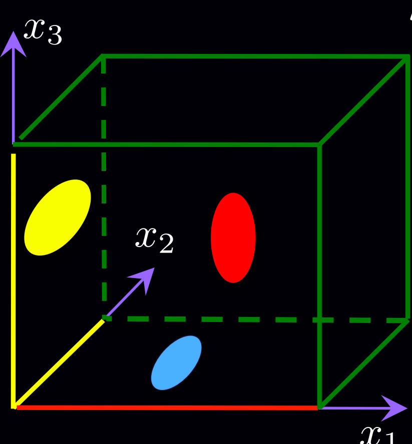

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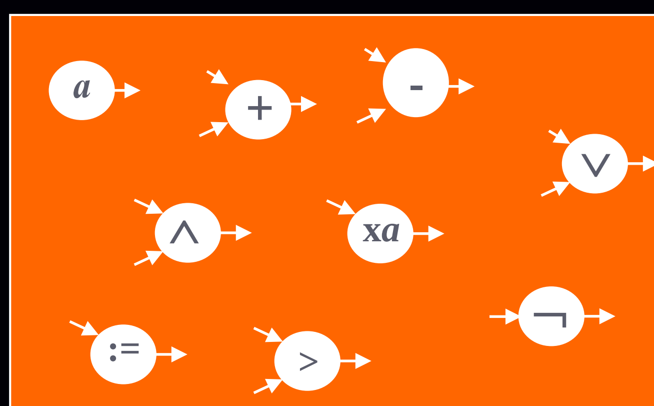

ARITHMCIRCUITSAT

NASH

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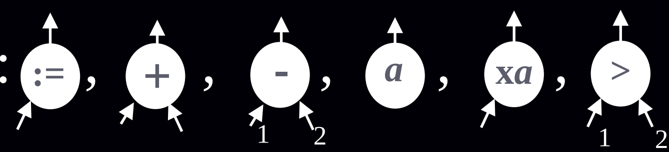
- ArithmCircuitSat

ArithmCircuitSat

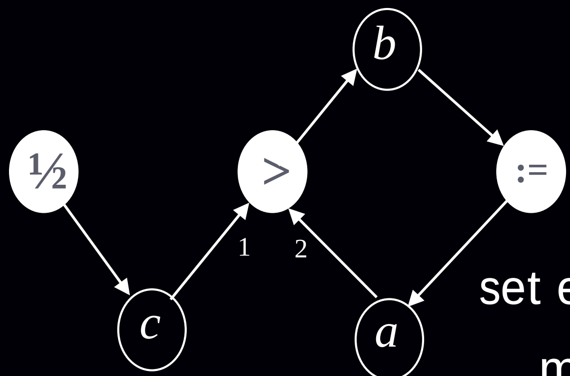
[Daskalakis, Goldberg, Papadimitriou'06]

INPUT: A circuit comprising:

- variable nodes $v_1,\ldots, v_n$
- gate nodes $g_1,\ldots, g_m$ of types: 

- directed edges connecting variables to gates and gates to variables (loops are allowed);
- variable nodes have in-degree 1; gates have 0, 1, or 2 inputs depending on type as above; gates & nodes have arbitrary fan-out

OUTPUT: Values $v_1,\ldots, v_n \in [0,1]$ satisfying the gate constraints:



assignment : $y == x_1$

addition : $y == \min\{1, x_1 + x_2\}$

subtraction : $y == \max\{0, x_1 - x_2\}$

set equal to a constant : $y == \max\{0, \min\{1, a\}\}$

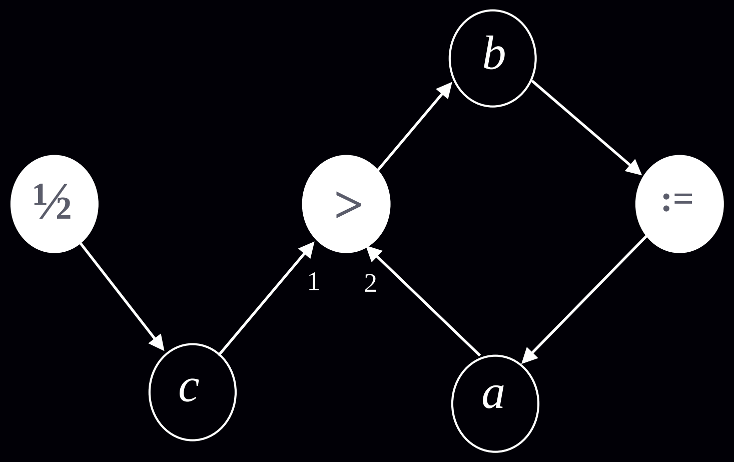
multiply by constant : $y == \max\{0, \min\{1, a \cdot x_1\}\}$

Comparator Gate Constraints

$$y == \begin{cases} 1, & \text{if } x_1 > x_2 \\ 0, & \text{if } x_1 < x_2 \\ *, & \text{if } x_1 = x_2 \end{cases}$$

any value is allowed

ARITHMCIRCUITSAT (example)



Satisfying assignment?

$a = b = c = ½$

ARITHMCIRCUITSAT

[Daskalakis, Goldberg, Papadimitriou’06]

INPUT: A circuit comprising:

- variable nodes $v_1, \ldots, v_n$
- gate nodes $g_1, \ldots, g_m$ of types: :=, +, - (inputs 1, 2), a, xa, > (inputs 1, 2)
- directed edges connecting variables to gates and gates to variables (loops are allowed);
- variable nodes have in-degree 1; gates have 0, 1, or 2 inputs depending on type as above; gates & nodes have arbitrary fan-out

OUTPUT: An assignment of values $v_1, \ldots, v_n \in [0,1]$ satisfying:

[DGP’06]: Always exists satisfying assignment!

[DGP’06]: but is PPAD-complete to find

:= $y == x_1$

\+ $y == \min\{1, x_1 + x_2\}$

\- $y == \max\{0, x_1 - x_2\}$

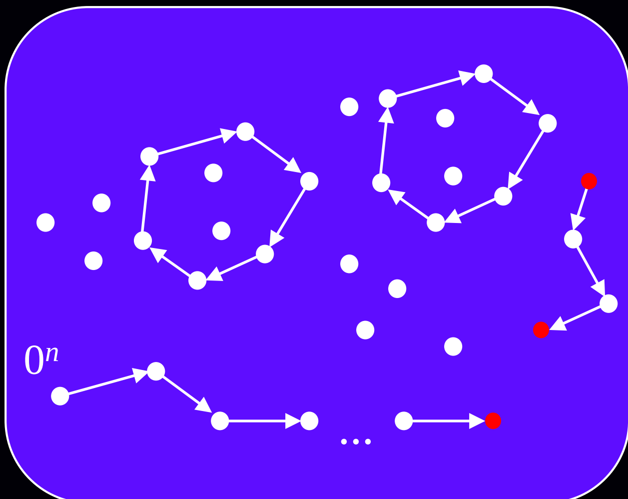
a $y == \max\{0, \min\{1, a\}\}$

xa $y == \max\{0, \min\{1, a \cdot x_1\}\}$

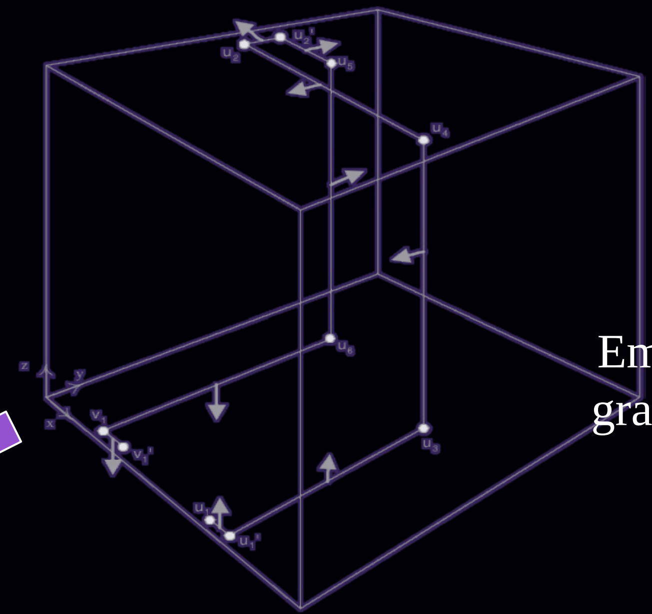
\> $$y == \begin{cases} 1, & \text{if } x_1 > x_2 \\ 0, & \text{if } x_1 < x_2 \\ *, & \text{if } x_1 = x_2 \end{cases}$$

PPAD-Completeness of NASH

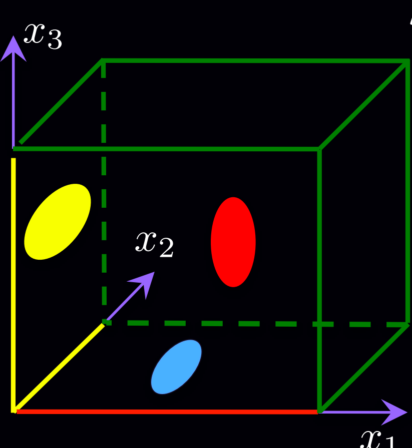
[Daskalakis, Goldberg, Papadimitriou’06]



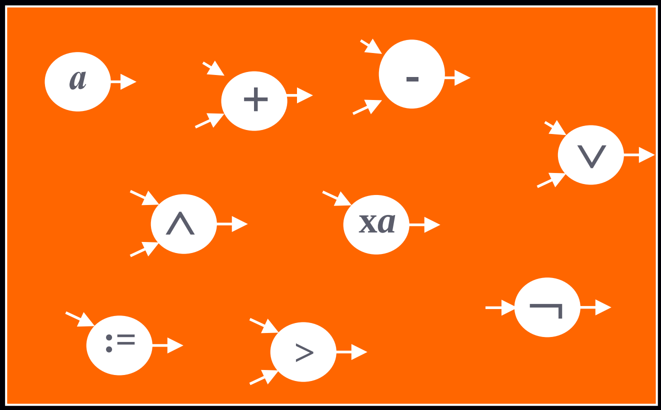
Generic PPAD



Embed PPAD graph in $[0,1]^3$



3D-SPERNER



ARITHMCIRCUITSAT

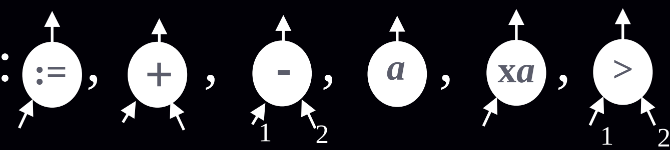
NASH

APPROXIMATE-ARITHMCIRCUITSAT

[Chen, Deng, Teng'06]

INPUT: 1. A circuit comprising:

- variable nodes $x_1, \dots, x_n$
- gate nodes $g_1, \dots, g_m$ of types:



- directed edges connecting variables to gates and gates to variables (loops are allowed);
- variable nodes have in-degree 1; gates have 0, 1, or 2 inputs depending on type as above; gates & nodes have arbitrary fan-out

2. $\varepsilon = 1/(n+m)^{\gamma}$, for some given $\gamma > 0$

OUTPUT: An assignment of values $x_1, \dots, x_n \in [0,1]$ satisfying:

[CDT'06]: still PPAD-complete to find

:= $y == x_1 \pm \epsilon$

\+ $y == \min\{1, x_1 + x_2\} \pm \epsilon$

\- $y == \max\{0, x_1 - x_2\} \pm \epsilon$

a $y == \max\{0, \min\{1, a\}\} \pm \epsilon$

xa $y == \max\{0, \min\{1, a \cdot x_1\}\} \pm \epsilon$

\> $$y == \begin{cases} 1, & \text{if } x_1 > x_2 + \epsilon \\ 0, & \text{if } x_1 < x_2 + \epsilon \\ *, & \text{if } x_1 = x_2 \pm \epsilon \end{cases}$$

Menu

- Existence Theorems: Nash, Brouwer, Sperner
- Total Search Problems in NP
- Identifying the Combinatorial Core
- Litmus Test: PPAD-completeness Results
- ARITHMCIRCUITSAT
- PPAD-completeness of:
  - Nash, Market Equilibrium,
  - Fractional Hypergraph matching, Scarf's Lemma

NEXT TIME: Other existence arguments: PPA, PPP, PLS