6.5310: Geometric Folding Algorithms, Spring 2025
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Problem Set 5 Solutions

Due: Tuesday, March 11, 2025 at 10am

Problem 5.1 [Tube Folding]. Ben Boxdiddle wants to fold a $1 \times 1 \times n$ box (Figure 1) from a square piece of paper. Ben can fold only along horizontal, vertical, and 45° diagonal lines, and only between points with integer or half-integer coordinates (an ancient tradition of the Boxdiddle family). He is considering several different approaches:

(a) Use the cube extrusion method from Lecture 7.

(b) Fold the paper into a width-1 rectangular strip and wrap it around the polycube.

(c) Do something else? Find the best method you can think of.

For each of these approaches, do the following:

- calculate the exact size of the smallest square that Ben needs; and
- estimate the number of edges in the crease pattern (an answer like $\Theta(n^3)$ is sufficient).

Which approach should Ben use?

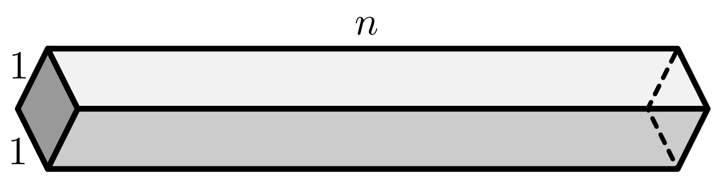

Figure 1: $1 \times 1 \times n$ box

Solution:

(a) **Cube Extrusion.** There are a few possible answers here, depending on exactly how you choose to implement the cube extrusion method (one step of which is Figure 2).

i. **The naïve way ($4n + 2$)**: To make each cube, we need to use another cube gadget. Each cube gadget consumes 4 columns and 2 rows of paper, so to make n cubes we need to consume $4n$ columns and $2n$ rows. The middle cube gadget also needs 1 column, and we need 1 final column to close the bottom of the tube. Figure 3 shows what 2 layers looks like.

Each cube gadget needs $O(1)$ creases, but the intersections between different pleats causes the crease pattern to have $O(n^2)$ edges.

ii. **Switching directions ($3n + 2$ or $3n + 3$ depending on parity)**: We notice that in the naïve way, we use only $2n$ rows, but $4n$ columns of the paper. To be more efficient, we rotate half of the cube gadgets 90 degrees, so that each pair of cube gadgets consumes 6 rows and 6 columns.

Each cube gadget needs $O(1)$ creases, but the intersections between different pleats causes the crease pattern to have $O(n^2)$ edges.

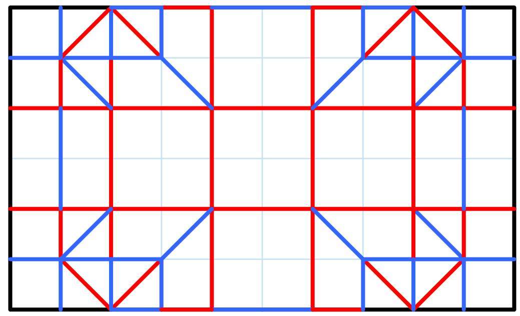

Figure 2: The basic cube extrusion crease pattern. The black box is the bounding box. There are 4 columns and two rows consumed by this pattern

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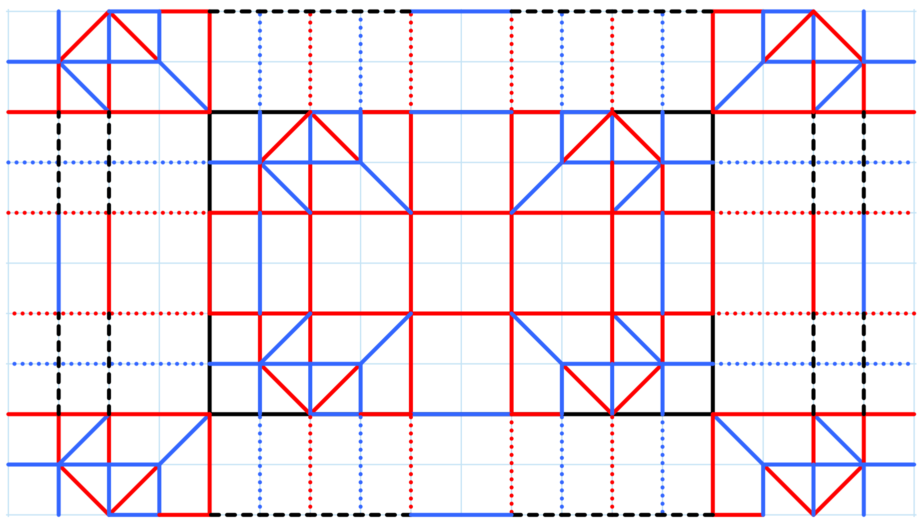

Figure 3: The basic cube extrusion crease pattern after 2 iterations. The dotted lines are the infinite rays that extend out of the inner cube gadget. The black dashed lines are creases of the outer gadget that go through the tucked material; they alternate M-V at each crease they intersect.

iii. **Pleat sharing ($3n$)**: If we tile n copies of the gadget stacked together (Figure 4 shows what $n = 2$ looks like) vertically, these will successfully all fold simultaneously by sharing the pleats that come out of the vertical sides. This needs a $3n \times 5$ sheet of paper, with one extra column needed to cover the bottom.

This method doesn't have the different pleats colliding like before, so it only uses $O(n)$ creases on the $3n \times 5$ sheet. However, we need to start from a square sheet, and folding that down to $3n \times 5$ gives us $O(n)$ layers of paper which all of these creases go through, causing the overall crease pattern to still have $O(n^2)$ edges.

iv. **Face sharing ($n + 2$)**: If we look at the pleat sharing method, we can see some inefficiency, where adjacent cubes have a pair of vertical faces sandwiched between them. A slight modification of the cube gadget allows two adjacent cubes that share a face to be fused together, with the crease pattern shown in Figure 5.

This method uses only $O(1)$ creases for the entire box; however like the previous method it needs to first fold down to a $O(n)$ layer thick sheet, so it actually has $O(n)$ edges.

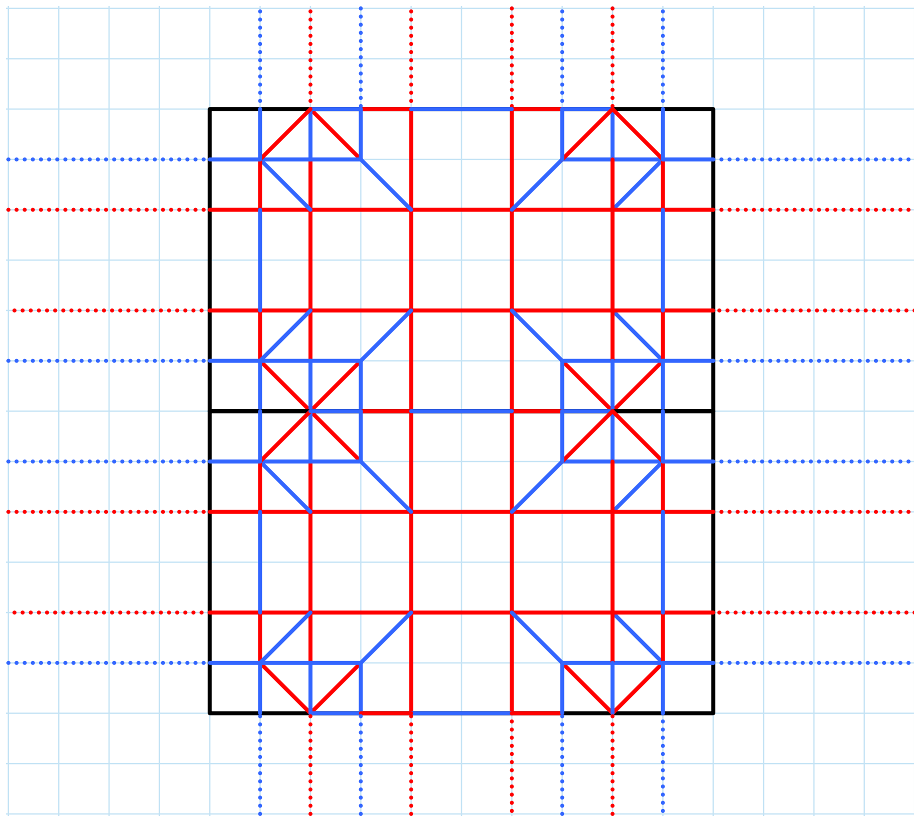

Figure 4: Two copies of the cube gadget sharing vertical pleats. The dotted lines are rays going out to infinity; if two cube gadgets are in the same row or column they can share these folds making the pattern more efficient in some cases.

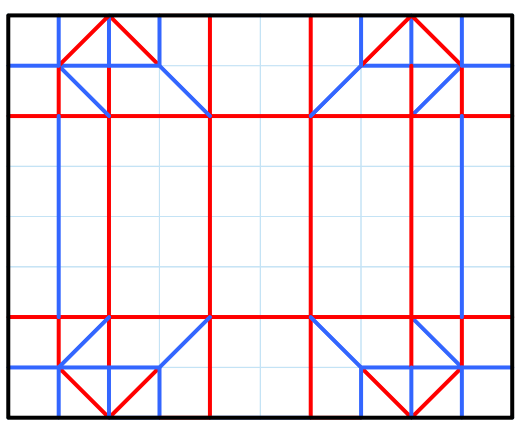

Figure 5: Two copies of the cube gadget sharing a vertical face. This can be extended to be n long.

The pleat sharing and face sharing methods are described more in Aviv's master's thesis.

(b) **Strip Folding.** There are two reasonable ways to strip fold this box: the long way and the short way. Both methods first need to fold the paper down to a strip; this causes every fold in the strip to go through $O(n)$ layers of paper and contribute $O(n)$ edges to the crease pattern.

i. **The Long Way.** Fold the strip along the long axis of the box for a length of n, then turn 90 degrees once on the top of the box, then go all the way along down and back up without turning (for a length of $2n + 1$), and finally turn 90 degrees again on top of the box to go down the last uncovered face. This uses $4n + 3$ of strip length. The crease pattern for $n = 3$ is in Figure 6.

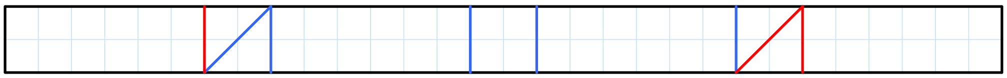

Figure 6: A crease pattern that covers the box by wrapping around the long way.

This method uses $O(1)$ folds, for $O(n)$ edges on the crease pattern.

ii. **The Short Way.** We wrap the box around and around the short way. Each unit of the box needs 5 units of strip: 3 to go around the box and 2 to turn up to the next layer. 2 more units are needed on the top and bottom, for $5n + 2$ total strip length. The crease pattern for $n = 3$ is in Figure 7.

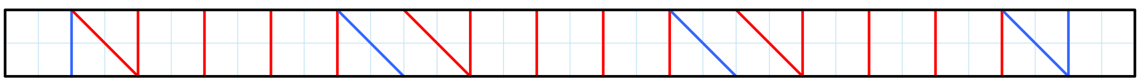

Figure 7: A crease pattern that covers the box by wrapping around the short way.

This method uses $O(n)$ folds, for $O(n^2)$ edges on the crease pattern.

(c) Something else:

Naturally there is a lot of things you can choose here, but Figure 8 is a simple way to get $n + 1$ sized paper with only $O(n)$ creases.

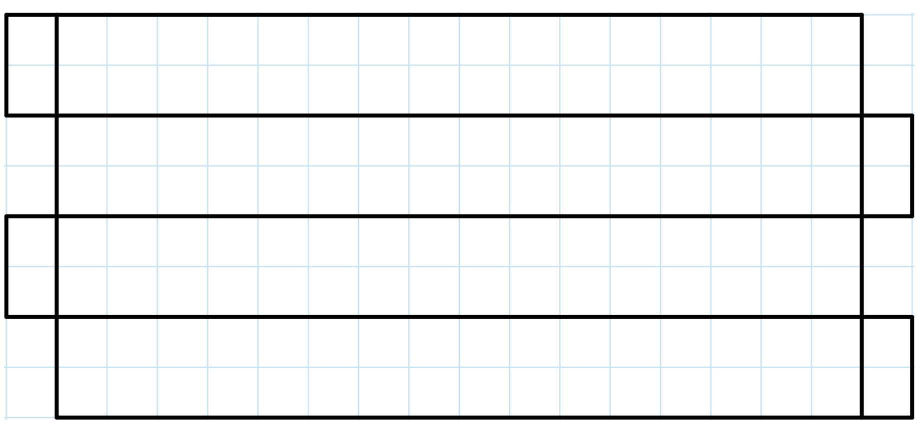

Figure 8: A net of the $1 \times 1 \times n$ box that fits in a $n + 1$ side length square.