Problem Set 1

This problem set is due **in recitation** on **Friday, February 13**.

Reading: Chapters §1, 2.1-2.3, 3, 4, 28.2, 30.1, Akra-Bazzi Handout

There are **five** problems. Each problem is to be done on a **separate sheet** (or sheets) of paper. Mark the top of each sheet with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated.

You will often be called upon to "give an algorithm" to solve a certain problem. Giving an algorithm entails:

1. A description of the algorithm in English and, if helpful, pseudocode.
2. A proof (or argument) of the correctness of the algorithm.
3. An analysis of the running time of the algorithm.

It is also suggested that you include at least one worked example or diagram to show more precisely how your algorithm works. Remember, your goal is to communicate. Graders will be instructed to take off points for convoluted and obtuse descriptions. If you cannot solve a problem, give a brief summary of any partial results.

Problem 1-1. Asymptotic Notation

Decide whether these statements are **True** or **False**. You must justify all your answers to receive full credit.

(a) $f(n) = \Omega(g(n)) \implies g(n) = O(f(n))$

(b) $f(n) = \omega(g(n)) \implies f(n) = \Omega(g(n))$

(c) $f(n) = O(g(n)) \wedge f(n) = \Omega(h(n)) \implies g(n) = \Theta(h(n))$

(d) $o(g(n)) \cap \omega(g(n)) = \emptyset$

(e) $f(n) = O(g(n)) \wedge g(n) = \Omega(f(n)) \implies f(n) = \Theta(g(n))$

Problem 1-2. More Asymptotic Notation

Rank the following functions by increasing order of growth; that is, find an arrangement $g_1, g_2, \ldots, g_{20}$ of the functions satisfying $g_1 = O(g_2)$, $g_2 = O(g_3)$, ..., $g_{19} = O(g_{20})$. Partition your list into equivalence classes such that $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$.

$\binom{n}{2}$	$n \log n$	$\sum_{k=1}^{n} \frac{1}{k}$	$8n^2$	$\log \sqrt{\log n}$
$n!$	$\log \log n$	$n^{\log n}$	$\log n!$	$4^{\log n}$
$\sum_{k=0}^{n} \binom{n}{k}$	$2^{\log^2 n}$	10^{100}	3^n	$\log n$
$(\sqrt{2})^{\log n}$	$(n-1)!$	$3n^3$	2^n	$5\sqrt{n}$

Problem 1-3. Recurrence Relations

Solve the following recurrences. Give a Θ bound for each problem. If you are unable to find a Θ bound, provide as tight upper (O or o) and lower (Ω or ω) bounds as you can find. Justify your answers. You may assume that $T(1) = O(1)$.

(a) $T(n) = T(\sqrt{n}) + 1$

(b) $T(n) = 2T(n-1) + 1$

(c) $T(n) = 2T(n/3) + 1$

(d) $T(n) = 49T(n/25) + (\sqrt{n})^3 \log n$

(e) $T(n) = 9T(n/3) + n^2 \log n$

(f) $T(n) = 8T(n/2) + n^3$

Problem 1-4. Divide and Conquer Multiplication

(a) Show how to multiply two linear polynomials $ax + b$ and $cx + d$ using only three multiplications. (Hint: One of the multiplications is $(a + b) \cdot (c + d)$.)

(b) Give a divide-and-conquer algorithm for multiplying two polynomials of degree-bound n that runs in time $\Theta(n^{\log 3})$.

(c) Show that two n-bit integers can be multiplied in $O(n^{\log 3})$ steps, where each step operates on at most a constant number of 1-bit values.

Problem 1-5. Finding a Pair that Sums to x

Give a $\Theta(n \log n)$ algorithm which, given a set S of n real numbers and another real number x, determines whether or not there exists two elements in S whose sum is exactly x.