Mathematics for Computer Science

MIT 6.042J/18.062J

More (Un)Countable Sets

Proving Uncountability

Lemma.

If U is an uncountable set and

A surj U,

then A is uncountable.

The Real Numbers are Uncountable

Decimal expansions:

$$\sqrt{2} = 1.4142\ldots, \qquad 5 = 5.000\ldots$$

$$\frac{1}{10} = 0.1000\ldots, \qquad \frac{1}{3} = 0.333\ldots$$

$$\frac{1}{99} = 0.010101\ldots$$

The Real Numbers are Uncountable

b(r) ::= 0,1 decimals of r

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The Real Numbers are Uncountable

$b(r)$ ::= 0,1 decimals of r

$b\left(\sqrt{2}\right)$ undefined, $\qquad b(5) = 000\ldots$

$b\left(\frac{1}{10}\right) = 1000\ldots,\qquad b\left(\frac{1}{3}\right)$ undefined,

$$b\left(\frac{1}{99}\right) = 010101\ldots$$

The Real Numbers are Uncountable

$$b:\mathbb{R}\rightarrow\{0,1\}^{\omega}$$

is a surjective function

The Real Numbers are Uncountable

$$\mathbb{R}\ \text{surj}\ \{0,1\}^{\omega}$$

So $\mathbb{R}$ is uncountable

Proving countability

Lemma.

If C is a countable set and

$$C\ \text{surj}\ A,$$

then A is countable.

Sequences of positive ints

Sequences of $\mathbb{Z}^+$

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e(n) ::= exponents of primes in the factorization of n

$$e(3^4\, 7^{22}\, 23^{11}) = (4, 22, 11)$$

Sequences of $\mathbb{Z}^+$

e(n) ::= exponents of primes in the factorization of n

$$e : \mathbb{N} \to (\mathbb{Z}^+)^*$$

is a surjective function

Sequences of $\mathbb{Z}^+$

$$\mathbb{N} \text{ surj } (\mathbb{Z}^+)^*$$

So $(\mathbb{Z}^+)^*$ is countable