

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

# More (Un)Countable Sets



Albert R Meyer, February 28, 2014

uncount.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Proving Uncountability

Lemma.

If  $U$  is an uncountable set and

$A \text{ surj } U,$

then  $A$  is uncountable.



Albert R Meyer, February 28, 2014

uncount.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Real Numbers are Uncountable

Decimal expansions:

$$\sqrt{2} = 1.4142\dots, \quad 5 = 5.000\dots$$

$$\frac{1}{10} = 0.1000\dots, \quad \frac{1}{3} = 0.333\dots$$

$$\frac{1}{99} = 0.010101\dots$$



Albert R Meyer, February 28, 2014

uncount.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Real Numbers are Uncountable

$b(r) ::= 0,1$  decimals of  $r$

$$\sqrt{2} = 1.4142\dots, \quad 5 = 5.000\dots$$

$$\frac{1}{10} = 0.1000\dots, \quad \frac{1}{3} = 0.333\dots$$

$$\frac{1}{99} = 0.010101\dots$$



Albert R Meyer, February 28, 2014

uncount.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### The Real Numbers are Uncountable

$b(r) ::= 0,1$  decimals of  $r$

$$\sqrt{2} = 1.4142\dots, \quad b(5) = 000\dots$$

$$\frac{1}{10} = 0.1000\dots, \quad \frac{1}{3} = 0.333\dots$$

$$\frac{1}{99} = 0.010101\dots$$



Albert R Meyer, February 28, 2014

uncount.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### The Real Numbers are Uncountable

$b(r) ::= 0,1$  decimals of  $r$

$$\sqrt{2} = 1.4142\dots, \quad b(5) = 000\dots$$

$$b\left(\frac{1}{10}\right) = 1000\dots, \quad \frac{1}{3} = 0.333\dots$$

$$\frac{1}{99} = 0.010101\dots$$



Albert R Meyer, February 28, 2014

uncount.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### The Real Numbers are Uncountable

$b(r) ::= 0,1$  decimals of  $r$

$$\sqrt{2} = 1.4142\dots, \quad b(5) = 000\dots$$

$$b\left(\frac{1}{10}\right) = 1000\dots, \quad \frac{1}{3} = 0.333\dots$$

$$b\left(\frac{1}{99}\right) = 010101\dots$$



Albert R Meyer, February 28, 2014

uncount.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### The Real Numbers are Uncountable

$b(r) ::= 0,1$  decimals of  $r$

$$b\left(\sqrt{2}\right) \text{ undefined,} \quad b(5) = 000\dots$$

$$b\left(\frac{1}{10}\right) = 1000\dots, \quad b\left(\frac{1}{3}\right) \text{ undefined,}$$

$$b\left(\frac{1}{99}\right) = 010101\dots$$



Albert R Meyer, February 28, 2014

uncount.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Real Numbers are Uncountable

$$b: \mathbb{R} \rightarrow \{0,1\}^\omega$$

is a surjective function



Albert R Meyer, February 28, 2014

uncount.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Real Numbers are Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^\omega$$

So  $\mathbb{R}$  is uncountable



Albert R Meyer, February 28, 2014

uncount.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Proving countability

Lemma.

If  $C$  is a countable set and

$$C \text{ surj } A,$$

then  $A$  is countable.



Albert R Meyer, February 28, 2014

uncount.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sequences of positive ints



Albert R Meyer, February 28, 2014

uncount.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sequences of $\mathbb{Z}^+$



Albert R Meyer, February 28, 2014

uncount.13

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sequences of $\mathbb{Z}^+$

$e(n) ::=$  exponents of primes in the factorization of  $n$

$$e(3^4 7^{22} 23^{11}) = (4, 22, 11)$$



Albert R Meyer, February 28, 2014

uncount.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sequences of $\mathbb{Z}^+$

$e(n) ::=$  exponents of primes in the factorization of  $n$

$$e: \mathbb{N} \rightarrow (\mathbb{Z}^+)^*$$

is a surjective function



Albert R Meyer, February 28, 2014

uncount.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sequences of $\mathbb{Z}^+$

$$\mathbb{N} \text{ surj } (\mathbb{Z}^+)^*$$

So  $(\mathbb{Z}^+)^*$  is countable



Albert R Meyer, February 28, 2014

uncount.16