


6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
 MIT 6.042J/18.062J


State Machines

 Albert R Meyer February 27, 2013 statemachine.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

State machines

step by step processes
 (may step in response
 to **input** –not today)

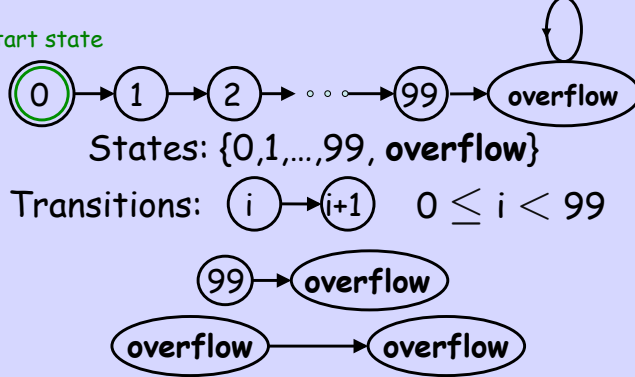
 Albert R Meyer February 27, 2013 statemachine.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

State machines

The **state graph** of a 99-bounded counter:


start state



States: $\{0, 1, \dots, 99, \text{overflow}\}$

Transitions: $i \rightarrow i+1 \quad 0 \leq i < 99$

$99 \rightarrow \text{overflow}$
 $\text{overflow} \rightarrow \text{overflow}$

 Albert R Meyer February 27, 2013 statemachine.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard



© Twentieth Century Fox. All rights reserved.


Picture source: <http://movieweb.com/movie/diehard3/>

 Albert R Meyer February 27, 2013 statemachine.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard

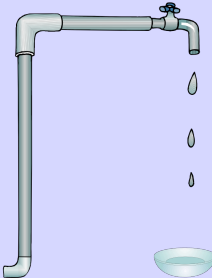
Simon says: On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

 Albert R Meyer February 27, 2013 statemachine.5


6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard


Supplies:




Water



3 Gallon Jug




5 Gallon Jug

 Albert R Meyer February 27, 2013 statemachine.6



6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard


Transferring water:



3 Gallon Jug




5 Gallon Jug

 Albert R Meyer February 27, 2013 statemachine.7

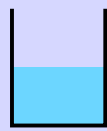

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard


Transferring water:



3 Gallon Jug



5 Gallon Jug

 Albert R Meyer February 27, 2013 statemachine.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die hard state machine

State:

amount of water in jugs: (b, l)

$$0 \leq b \leq 5, 0 \leq l \leq 3$$

Start State: $(0, 0)$

Albert R Meyer February 27, 2013 statemachine.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

State machines

Die Hard Transitions:

1. Fill little jug: $(b, l) \rightarrow (b, 3)$ for $l < 3$
2. Fill big jug: $(b, l) \rightarrow (5, l)$ for $b < 5$
3. Empty little jug: $(b, l) \rightarrow (b, 0)$ for $l > 0$
4. Empty big jug: $(b, l) \rightarrow (0, l)$ for $b > 0$

Albert R Meyer February 27, 2013 statemachine.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

State machines

5. Pour big jug into little jug
 - (i) If no overflow, then $(b, l) \rightarrow (0, b+l)$
 $b+l \leq 3$
 - (ii) otherwise $(b, l) \rightarrow (b-(3-l), 3)$
6. Pour little jug into big jug.
Likewise

Albert R Meyer February 27, 2013 statemachine.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard

Simon's challenge:

Disarm the bomb by putting precisely 4 gallons of water on the scale, or it will **blow up**.
(You can figure out how)

Albert R Meyer February 27, 2013 statemachine.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard

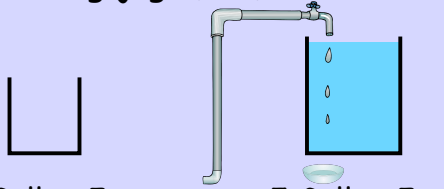
Work it out now!

Albert R Meyer February 27, 2013 statemachine.13

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

How to do it

Start with empty jugs: (0,0)
Fill the big jug: (5,0)



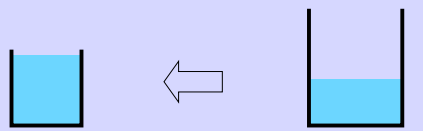
3 Gallon Jug 5 Gallon Jug

Albert R Meyer February 27, 2013 statemachine.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

How to do it

Pour from big to little: (2,3)




3 Gallon Jug 5 Gallon Jug

Albert R Meyer February 27, 2013 statemachine.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

How to do it

Empty the little: (2,0)




3 Gallon Jug 5 Gallon Jug

Albert R Meyer February 27, 2013 statemachine.16

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

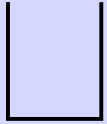
How to do it

Pour from big to little: (0,2)



3 Gallon Jug

←




5 Gallon Jug

Albert R Meyer February 27, 2013 statemachine.17

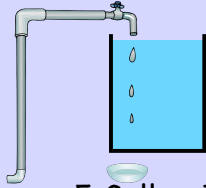
6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

How to do it

Fill the big jug: (5,2)



3 Gallon Jug




5 Gallon Jug

Albert R Meyer February 27, 2013 statemachine.18

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

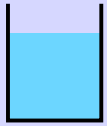
How to do it

Pour from big to little: (4,3)



3 Gallon Jug

←



5 Gallon Jug


Done!

Albert R Meyer February 27, 2013 statemachine.19

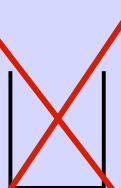
6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard once and for all

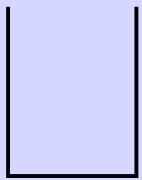
What if have a 9 gallon jug instead?



3 Gallon Jug



~~5 Gallon Jug~~



9 Gallon Jug

Can you do it? Can you prove it?

Albert R Meyer February 27, 2013 statemachine.20

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Preserved Invariants

Die hard once and for all
preserved invariant:

$P(\text{state}) ::= \text{"3 divides the number of gallons in each jug."}$

$P((b,l)) ::= (3 \mid b \text{ AND } 3 \mid l)$

Albert R Meyer February 27, 2013 statemachine.22

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Preserved Invariants

Die hard once and for all
preserved invariant:

$(b,l) \rightarrow (b-(3-l),3)$

$P((b,l)) ::= (3 \mid b \text{ AND } 3 \mid l)$

Albert R Meyer February 27, 2013 statemachine.23

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Die Hard Once & For All

Corollary: No state $(4,x)$ is reachable, so
Bruce Dies!

Albert R Meyer February 27, 2013 statemachine.24

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Floyd's Invariant Principle

(induction for state machines)

Preserved Invariant, $P(\text{state})$:

if $P(q)$ and $q \rightarrow r$, then $P(r)$

Conclusion: if $P(\text{start})$, then $P(r)$
for all reachable states r ,
including final state (if any)

Albert R Meyer February 27, 2013 statemachine.25

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Diagonal Robot

the robot is on a grid

Albert R Meyer February 27, 2013 statemachine.27

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Diagonal Robot

it can **move diagonally**

Albert R Meyer February 27, 2013 statemachine.28

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Diagonal Robot

can it get from (0,0) to (1,0)?

Albert R Meyer February 27, 2013 statemachine.29

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Robot Preserved Invariant

NO! preserved invariant:
 $P((x, y)) ::= x + y$ is even
 move adds ± 1 to **both** x & y ,
 preserving parity of $x+y$.
 Also, $P((0, 0))$ is true.

Albert R Meyer February 27, 2013 statemachine.30

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Robot Preserved Invariant

So in all positions (x,y)
reachable from $(0,0)$,
 $x + y$ stays **even**
But $1 + 0 = 1$ is odd, so
 $(1,0)$ is **not reachable**

Albert R Meyer February 27, 2013 statemachine.31

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Fifteen Puzzle Explained!

--by similar reasoning
details in problem 2

Albert R Meyer February 27, 2013 statemachine.32

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Fast Exponentiation

compute a^b using registers X, Y, Z, R

$X := a; Y := 1; Z := b;$

REPEAT:

if $Z=0$, then return Y

$R := \text{remdr}(Z, 2); Z := \text{quotnt}(Z, 2)$

if $R=1$, then $Y := X \cdot Y$

$X := X^2$

Albert R Meyer February 27, 2013 statemachine.33

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Fast Exponentiation

State Machine:

States ::= $\mathbb{R} \times \mathbb{R} \times \mathbb{N}$

start ::= $(a, 1, b)$

transitions ::= $(X, Y, Z) \rightarrow$
 $(X^2, Y, \text{quotnt}(Z, 2))$ if $Z > 0$ is even
 $(X^2, X \cdot Y, \text{quotnt}(Z, 2))$ if $Z > 0$ is odd

Albert R Meyer February 27, 2013 statemachine.34

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Fast Exponentiation

Preserved Invariant: $YX^Z = a^b$
 $(X, Y, Z) \rightarrow [Z > 0 \text{ is odd}]$
 $(X^2, X \cdot Y, (Z-1)/2)$

$$(X \cdot Y)(X^2)^{(Z-1)/2} = (X \cdot Y)X^{Z-1}$$

$$= YX^Z = a^b$$

Albert R Meyer February 27, 2013 statemachine.35

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Partial Correctness

preserved invariant: $YX^Z = a^b$
 at start $1 \cdot a^b = a^b$

at end $Z=0$, so return
 $Y = YX^0 = a^b$

Albert R Meyer February 27, 2013 statemachine.36

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2


Fast Termination

at each transition
 $Z := \text{quotient}(Z, 2)$
 $Z = b$ at start, so $Z = 0$
 in $\leq \log_2(b)$ transitions

Albert R Meyer February 27, 2013 statemachine.37

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Robert W Floyd (1934–2001)



Eulogy by Knuth: <http://www.acm.org/pubs/membernet/stories/floyd.pdf>
 Picture source: <http://www.stanford.edu/dept/news/report/news/november7/floydobit-117.html>

Albert R Meyer February 27, 2013 statemachine.38