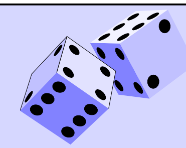
Mathematics for Computer Science
MIT 6.042J/18.062J

Great Expectations

Carnival Dice

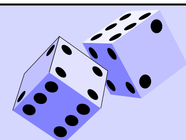


Choose a number from 1 to 6, then roll 3 fair dice:

win $1 for each match

lose $1 if no match

Carnival Dice



Example: choose 5, then

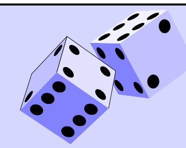
roll 2,3,4: lose $1

roll 5,4,6: win $1

roll 5,4,5: win $2

roll 5,5,5: win $3

Carnival Dice



Is this a fair game?

Carnival Dice

$$\Pr[0\ \text{fives}] = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$\Pr[1\ \text{five}] = \binom{3}{1}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)$$

$$\Pr[2\ \text{fives}] = \binom{3}{2}\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^2$$

$$\Pr[3\ \text{fives}] = \left(\frac{1}{6}\right)^3$$

Carnival Dice

# matches	probability	$ won
0	125/216	-1
1	75/216	1
2	15/216	2
3	1/216	3

Carnival Dice

so every 216 games, expect
0 matches about 125 times
1 match about 75 times
2 matches about 15 times
3 matches about once

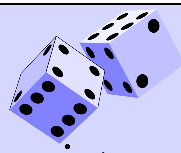
Carnival Dice

So on average expect to win:

$$\frac{125\cdot(-1) + 75\cdot 1 + 15\cdot 2 + 1\cdot 3}{216}$$

$$= -\frac{17}{216} \approx -8\,\text{cents}$$

Carnival Dice



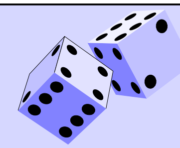
So on average expect to win:

$125 \cdot$ … $1 \cdot 3$

NOT fair!

$$= -\frac{17}{216} \approx -8\,\text{cents}$$

Carnival Dice



You can "expect" to lose 8 cents per play. But you never actually lose 8 cents on any single play, this is just your average loss.

Expected Value

The expected value of random variable R is the average value of R --with values weighted by their probabilities

Expected Value

The expected value of random variable R is

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

so $E[\$\text{win in Carnival}] = -\frac{17}{216}$

Alternative definition

$$E[R] = \sum_{\omega \in S} R(\omega) \cdot \Pr[\omega]$$

this form helpful in some proofs

Alternative definition

$$E[R] = \sum_{\omega \in S} R(\omega) \cdot \Pr[\omega]$$

proof of equivalence:

$[R = v] ::= \{\omega \mid R(\omega) = v\}$ so

$$\Pr[R = v] ::= \sum_{\omega \in [R=v]} \Pr[\omega]$$

proof of equivalence

Now

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$$= \sum_{v} \sum_{\omega \in [R=v]} R(\omega) \cdot Pr[\omega]$$

$$= \sum_{\omega \in S} R(\omega) \cdot Pr[\omega]$$

Sums vs Integrals

We get away with sums instead of integrals because the sample space is assumed countable:

$$\mathcal{S} = \{\omega_0, \omega_1, \ldots, \omega_n, \ldots\}$$

Rearranging Terms

It's safe to rearrange terms in sums because

Rearranging Terms

It's safe to rearrange terms in sums because we implicitly assume that the defining sum for the expectation is

absolutely convergent

Absolute convergence

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

the terms on the right could be rearranged to equal anything at all when the sum is not absolutely convergent

Expected Value

also called

mean value, mean, or

expectation

Expectations & Averages

From a pile of graded exams, pick one at random, and let S be its score.

Expectations & Averages

From a pile of graded exams, pick one at random, and let S be its score. Now E[S] equals

the average exam score

Expectations & Averages

We can estimate averages by estimating expectations of random variables

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We can estimate averages by estimating expectations of random variables based on picking random elements

sampling

Expectations & Averages

For example, it is impossible for all exams to be above average (no matter what the townspeople of Lake Woebegone say):

$$\Pr[R > E[R]] < 1$$

Expectations & Averages

On the other hand

$$\Pr[R > E[R]] \geq 1 - \varepsilon$$

is possible for all $\varepsilon > 0$

For example, almost everyone has an above average number of fingers.