

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Mathematics for Computer Science
MIT 6.042J/18.062J

Great Expectations



Albert R Meyer,

May 8, 2013

expect_intro.1

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Carnival Dice



Choose a number from 1 to 6,
then roll 3 fair dice:

win \$1 for each match

lose \$1 if no match



Albert R Meyer,

May 8, 2013

expect_intro.3

6	9	13	7
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Carnival Dice



Example: choose 5, then

roll 2,3,4: lose \$1

roll 5,4,6: win \$1

roll 5,4,5: win \$2

roll 5,5,5: win \$3



Albert R Meyer,

May 8, 2013

expect_intro.4

6	9	13	7
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Carnival Dice



Is this a
fair game?



Albert R Meyer,

May 8, 2013

expect_intro.5

6	9	13	7
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Carnival Dice



$$\Pr[0 \text{ fives}] = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$\Pr[1 \text{ five}] = \binom{3}{1} \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$$

$$\Pr[2 \text{ fives}] = \binom{3}{2} \left(\frac{5}{6}\right)^1 \left(\frac{1}{6}\right)^2$$

$$\Pr[3 \text{ fives}] = \left(\frac{1}{6}\right)^3$$



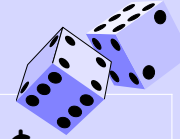
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May 8, 2013

expect_intro.6

6	9	13	7
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Carnival Dice



# matches	probability	\$ won
0	125/216	-1
1	75/216	1
2	15/216	2
3	1/216	3



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May 8, 2013

expect_intro.7

6	9	13	7
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Carnival Dice



so every 216 games, expect

- 0 matches about 125 times
- 1 match about 75 times
- 2 matches about 15 times
- 3 matches about once



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expect_intro.8

6	9	13	7
12	10	5	
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Carnival Dice



So on average expect to win:

$$\frac{125 \cdot (-1) + 75 \cdot 1 + 15 \cdot 2 + 1 \cdot 3}{216}$$

$$= -\frac{17}{216} \approx -8 \text{ cents}$$



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May 8, 2013

expect_intro.9

6	9	13	7
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Carnival Dice



So on average expect to win:

$$\frac{125 \cdot \boxed{\text{NOT fair!}} \cdot 3}{216} = -\frac{17}{216} \approx -8 \text{ cents}$$



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expect_intro.10

6	9	13	7
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Carnival Dice



You can "expect" to lose **8 cents** per play. But you **never actually** lose **8 cents** on any single play, this is just your **average loss**.



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expect_intro.11

6	9	13	7
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Expected Value

The **expected value** of random variable **R** is the **average value** of **R** --with values weighted by their probabilities



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expect_intro.12

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Expected Value

The **expected value** of random variable **R** is

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

$$\text{so } E[\text{\$win in Carnival}] = -\frac{17}{216}$$



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expect_intro.13

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Alternative definition

$$E[R] = \sum_{\omega \in \mathcal{S}} R(\omega) \cdot \Pr[\omega]$$

this form helpful in
some proofs



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expect_intro.14

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Alternative definition

$$E[R] = \sum_{\omega \in \mathcal{S}} R(\omega) \cdot \Pr[\omega]$$

proof of equivalence:

$$[R = v] ::= \{\omega \mid R(\omega) = v\} \text{ so}$$

$$\Pr[R = v] ::= \sum_{\omega \in [R=v]} \Pr[\omega]$$



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expect_intro.15

6	9	13	7
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proof of equivalence

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$



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expect_intro.16

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proof of equivalence

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$



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expect_intro.17

6	9	13	7
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proof of equivalence

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega]$$



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expect_intro.18

6	9	13	7
12	10	5	
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proof of equivalence

Now

$$\begin{aligned} E[R] & ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega] \\ & = \sum_v \sum_{\omega \in [R=v]} v \cdot \text{Pr}[\omega] \end{aligned}$$



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expect_intro.19

6	9	13	7
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proof of equivalence

Now

$$\begin{aligned} E[R] & ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega] \\ & = \sum_v \sum_{\omega \in [R=v]} v \cdot \text{Pr}[\omega] \end{aligned}$$



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expect_intro.20

6	9	13	7
12	10	5	
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proof of equivalence

Now

$$\begin{aligned} E[R] & ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega] \\ & = \sum_v \sum_{\omega \in [R=v]} R(\omega) \cdot \text{Pr}[\omega] \\ & = \sum_{\omega \in S} R(\omega) \cdot \text{Pr}[\omega] \end{aligned}$$



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expect_intro.21

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Sums vs Integrals

We get away with sums instead of integrals because the sample space is assumed **countable**:

$$\mathcal{S} = \{\omega_0, \omega_1, \dots, \omega_n, \dots\}$$



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expect_intro.23

6	9	13	7
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Rearranging Terms

It's safe to rearrange terms in sums because



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expect_intro.24

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Rearranging Terms

It's safe to rearrange terms in sums because we implicitly assume that the defining sum for the expectation is **absolutely convergent**



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expect_intro.25

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Absolute convergence

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

the terms on the right could be rearranged to equal anything at all when the sum is **not** absolutely convergent



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expect_intro.26

6	9	13	7
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Expected Value

also called
mean value, mean, or
expectation



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May 8, 2013

expect_intro.27

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Expectations & Averages

From a pile of graded exams,
pick one at random, and let S
be its score.



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May 8, 2013

expect_intro.28

6	9	13	7
12		10	5
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Expectations & Averages

From a pile of graded exams,
pick one at random, and let S
be its score. Now $E[S]$ equals
the average exam score



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May 8, 2013

expect_intro.29

6	9	13	7
12		10	5
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Expectations & Averages

We can estimate averages
by estimating expectations
of random variables



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May 8, 2013

expect_intro.30

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Expectations & Averages

We can estimate **averages** by estimating **expectations** of random variables based on picking random elements

sampling



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expect_intro.31

6	9	13	7
12		10	5
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Expectations & Averages

For example, it is impossible for all exams to be above average (no matter what the townspeople of Lake Woebegone say):

$$\Pr[R > E[R]] < 1$$



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May 8, 2013

expect_intro.32

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Expectations & Averages

On the other hand

$$\Pr[R > E[R]] \geq 1 - \epsilon$$

is possible for all $\epsilon > 0$

For example, almost everyone has an above average number of fingers.



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expect_intro.33