Problem Set 8

Due: May 11

Reading:

- Chapter 17. Events and Probability Spaces

Problem 1. (a) Show that any odd integer \( x \) in the range \( 10^9 < x < 2 \cdot 10^9 \) containing all ten digits \( 0, 1, \ldots, 9 \) must have consecutive even digits.

Hint: What can you conclude about the parities of the first and last digit?

(b) Show that there are 2 vertices of equal degree in any finite undirected graph with \( n \geq 2 \) vertices.

Hint: Cases conditioned upon the existence of a degree zero vertex.

Problem 2. (a) Let \( S = \{1, 2, 3, 4\}^n \) be the set of length-\( n \) sequences \( (a_1, \ldots, a_n) \) where each \( a_j \) is chosen from \( \{1, 2, 3, 4\} \). What is \( |S| \)?

(b) How many of the sequences in \( S \) contain each of 1, 2, 3, and 4 at least once? Use Inclusion-Exclusion to find and prove your answer.

Hint: For \( 1 \leq i \leq 4 \), define \( S_i \subset S \) as the subset of sequences that do not contain \( i \). What is \( |S_1| \)? How about \( |S_1 \cap S_2| \)?

Problem 3.

The results of a round robin tournament in which every two people play each other and one of them wins can be modelled a tournament digraph—a digraph with exactly one directed edge between each pair of distinct vertices. We’ll draw a directed edge \( (v \rightarrow w) \) if player \( v \) beats player \( w \), and otherwise we’ll include directed edge \( (w \rightarrow v) \).

An \( n \)-player tournament is \( k \)-neutral for some \( k \in [0, n) \), when, for every set of \( k \) players, there is another player who beats them all. For example, being 1-neutral is the same as not having a “best” player who beats everyone else.

This problem will prove the existence of an \( n \)-player tournament that is 10-neutral, if \( n \) is large enough. We will do this by reformulating the question in terms of probabilities. In particular, for any fixed \( n \), we assign probabilities to each \( n \)-vertex tournament digraph by choosing a direction for the edge between any two vertices, independently and with equal probability for each edge.

(a) For any set \( S \) of 10 players, let \( B_S \) be the event that no contestant beats everyone in \( S \). Express \( \Pr[B_S] \) in terms of \( n \).

---

2018, Albert R Meyer. This work is available under the terms of the Creative Commons Attribution-ShareAlike 3.0 license.
(b) Let $Q$ be the event that the tournament digraph is not 10-neutral. Prove that
\[
\Pr[Q] \leq \binom{n}{10} \alpha^{n-10},
\]
where $\alpha := 1 - (1/2)^{10}$.
*Hint:* Let $S$ range over the size-10 subsets of players, so
\[
Q = \bigcup S.
\]
Use Boole’s inequality.

(c) Conclude that if $n$ is large enough, then $\Pr[Q] < 1$.
*Hint:* Show that the limit as $n$ approaches infinity is 0. Why is this sufficient?

(d) Explain why the previous result implies that there is an $n$-player 10-neutral tournament (for a large enough $n \in \mathbb{N}$).