Problem Set 6

Due: April 20

Reading: Chapter 9 through 9.10: *GCDs, Congruences, and Euler's Theorem*

Problem 1.
Here is a game you can analyze with number theory and always beat me. We start with two distinct, positive integers written on a blackboard. Call them a and b. Now we take turns. (I'll let you decide who goes first.) On each turn, the player must write a new positive integer on the board that is the difference of two numbers that are already there. If a player cannot play, then they lose.

For example, suppose that 12 and 15 are on the board initially. Your first play must be 3, which is $15-12$. Then I might play 9, which is $12-3$. Then you might play 6, which is $15-9$. Then I can't play, so I lose.

(a) Show that every number on the board at the end of the game is a multiple of $\gcd(a,b)$.

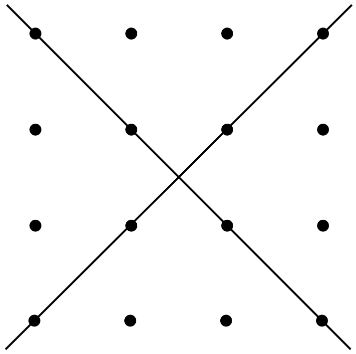
(b) Show that every positive multiple of $\gcd(a,b)$ up to $\max(a,b)$ is on the board at the end of the game.

(c) Describe a strategy that lets you win this game every time.

Problem 2.
Two nonparallel lines in the real plane intersect at a point. Algebraically, this means that the equations

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

have a unique solution (x, y), provided $m_1 \neq m_2$. This statement would be false if we restricted x and y to the integers, since the two lines could cross at a noninteger point:



However, an analogous statement holds if we work over the integers *modulo a prime* p. Find a solution to the congruences

$$y \equiv m_1 x + b_1 \pmod{p}$$
$$y \equiv m_2 x + b_2 \pmod{p}$$

when $m_1 \not\equiv m_2 \pmod{p}$. Express your solution in the form $x \equiv ? \pmod{p}$ and $y \equiv ? \pmod{p}$ where the ?'s denote expressions involving m_1, m_2, b_1 and b_2. You may find it helpful to solve the original equations over the reals first.

Problem 3.
In this problem we'll prove that for all integers a, m where $m > 1$,

$$a^m \equiv a^{m-\phi(m)} \pmod{m}. \tag{1}$$

Note that a and m need not be relatively prime.

Assume $m = p_1^{k_1} \cdots p_n^{k_n}$ for distinct primes, $p_1, \ldots, p_n$ and positive integers $k_1, \ldots, k_n$.

(a) Show that if p_i does not divide a, then

$$a^{\phi(m)} \equiv 1 \pmod{p_i^{k_i}}.$$

(b) Show that if $p_i \mid a$ then

$$a^{m-\phi(m)} \equiv 0 \pmod{p_i^{k_i}}. \tag{2}$$

(c) Conclude (1) from the facts above.

Hint: $a^m - a^{m-\phi(m)} = a^{m-\phi(m)}(a^{\phi(m)} - 1)$.