Problem Set 5

Due: April 6

Reading: Chapter 12. *Simple Graphs* through 12.8. *Connectivity*, omitting 12.6. *Coloring*.

Problem 1.
An edge is said to *leave* a set of vertices if one end of the edge is in the set and the other end is not.

(a) An n-node graph is said to be *mangled* if there is an edge leaving every set of $\lfloor n/2 \rfloor$ or fewer vertices. Prove the following:

Claim. *Every mangled graph is connected.*

An n-node graph is said to be *tangled* if there is an edge leaving every set of $\lceil n/3 \rceil$ or fewer vertices.

(b) Draw a tangled graph that is not connected.

(c) Find the error in the bogus proof of the following:

False Claim. *Every tangled graph is connected.*

Bogus proof. The proof is by strong induction on the number of vertices in the graph. Let $P(n)$ be the proposition that if an n-node graph is tangled, then it is connected. In the base case, $P(1)$ is true because the graph consisting of a single node is trivially connected.

For the inductive case, assume $n \geq 1$ and $P(1), \ldots, P(n)$ hold. We must prove $P(n+1)$, namely, that if an $(n+1)$-node graph is tangled, then it is connected.

So let G be a tangled, $(n+1)$-node graph. Choose $\lceil n/3 \rceil$ of the vertices and let G_1 be the tangled subgraph of G with these vertices and G_2 be the tangled subgraph with the rest of the vertices. Note that since $n \geq 1$, the graph G has a least two vertices, and so both G_1 and G_2 contain at least one vertex. Since G_1 and G_2 are tangled, we may assume by strong induction that both are connected. Also, since G is tangled, there is an edge leaving the vertices of G_1 which necessarily connects to a vertex of G_2. This means there is a path between any two vertices of G: a path within one subgraph if both vertices are in the same subgraph, and a path traversing the connecting edge if the vertices are in separate subgraphs. Therefore, the entire graph G is connected. This completes the proof of the inductive case, and the Claim follows by strong induction. ■

Problem 2.
Marvel is staging 4 test screenings of *Avengers: ∞ War* exclusively for a random selection of MIT students![1] For scheduling purposes, each of the selected students will specify which of the four screenings don't conflict with their schedule–every student is available for at least two out of the four screenings. However, each screening has only 20 available seats, not all of which need to be filled each time. Marvel is thus faced with a difficult scheduling problem: how do they make sure each of the chosen students is able to find a seat at a screening? They've recruited you to help solve this dilemma.

[1] Sadly this isn't actually happening, as far as we know.

(a) Describe how to model this situation as a matching problem. Be sure to specify what the vertices/edges should be and briefly describe how a matching would determine seat assignments for each student in a screening for which they are available. (This is a *modeling problem*; we aren't looking for a description of an algorithm to solve the problem.)

(b) Suppose 41 students have been selected. Can you guarantee that a matching exists, or are there some situations where not all of the 41 students can be accommodated? Briefly explain.

(c) If instead only 40 students are chosen, prove that there is always a matching.

Hint: Use Hall's Theorem or something similar. Is your graph degree constrained?