Problem Set 3

Due: March 9

Reading:

- Chapter 6. *State Machines*
- Chapter 7. *Recursive Data*

Problem 1.
Start with 102 coins on a table, 98 showing heads and 4 showing tails. There are two ways to change the coins:

(i) flip over any ten coins, or

(ii) let n be the number of heads showing. Place $n+1$ additional coins, all showing tails, on the table.

For example, you might begin by flipping nine heads and one tail, yielding 90 heads and 12 tails, then add 91 tails, yielding 90 heads and 103 tails.

(a) Model this situation as a state machine, carefully defining the set of states, the start state, and the possible state transitions.

(b) Explain how to reach a state with exactly one tail showing.

(c) Define the following derived variables:[1]

C	$::=$	the number of coins on the table,	H	$::=$	the number of heads,
T	$::=$	the number of tails,	C_2	$::=$	$\text{parity}(C)$,
H_2	$::=$	$\text{parity}(H)$,	T_2	$::=$	$\text{parity}(T)$.

Which of these variables is

1. strictly increasing?
2. weakly increasing?
3. strictly decreasing?
4. weakly decreasing?
5. constant?

No need to prove these answers.

(d) Prove using the Invariant principle that it is not possible to reach a state in which there is exactly one head showing. If you wish to use a fact from part c, please prove it carefully here.

[1]The function parity : $\mathbb{Z} \to \{0, 1\}$ is defined as follows: $\text{parity}(n) = 0$ when n is even, and $\text{parity}(n) = 1$ when n is odd.

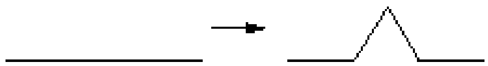

Figure 1 Constructing the Koch Snowflake.

Problem 2.
Give an example of a stable matching between 3 boys and 3 girls where no person gets their first choice. Briefly explain why your matching is stable. Can your matching be obtained from the Mating Ritual or the Ritual with boys and girls reversed?

Problem 3.
Fractals are an example of mathematical objects that can be defined recursively. In this problem, we consider the Koch snowflake. Any Koch snowflake can be constructed by the following recursive definition.

- **Base case**: An equilateral triangle with a positive integer side length is a Koch snowflake.
- **Constructor case**: Let K be a Koch snowflake, and let l be a single edge of the snowflake. Remove the middle third of l, and replace it with two line segments of the same length as the middle third, as shown in Figure 1

  The resulting figure is also a Koch snowflake.

(a) Find a single Koch snowflake that has exactly 9 edges and includes at least 3 different edge lengths.

(b) Prove using structural induction that the area inside any Koch snowflake is of the form $q\sqrt{3}$, where q is a rational number. Be sure to clearly label your induction hypothesis and other necessary assumptions during your proof.

Hint: If you require other facts about Koch snowflakes, be sure to prove those by structural induction too.