Problem Set 1

Due: February 16

Reading:

- Chapter 1. *What is a Proof?*
- Chapter 3. *Logical Formulas* through 3.5

These assigned readings do **not** include the Problem sections. (Many of the problems in the text will appear as class or homework problems.)

Reminder:

- Instructions for PSet submission are on the class Stellar page. Remember that each problem should prefaced with a *collaboration statement*.
- The class has a Piazza forum. With Piazza you may post questions—both administrative and content related—to the entire class or to just the staff. You are likely to get faster response through Piazza than from direct email to staff.

  You should post a question or comment to Piazza at least once by the end of the second week of the class; after that Piazza use is optional.

Problem 1.
Here is a generalization of Problem 1.16 that you may not have thought of:

Lemma. *Let the coefficients of the polynomial*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + x^m$$

be integers. Then any real root of the polynomial is either integral or irrational.

(a) Explain why the Lemma immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an mth power of some integer.

(b) Carefully prove the Lemma.

You may find it helpful to appeal to:
Fact. If a prime p is a factor of some power of an integer, then it is a factor of that integer.

You may assume this Fact without writing down its proof, but see if you can explain why it is true.

Problem 2. **(a)** Suppose that

$$a + b + c = d,$$

where a, b, c, d are nonnegative integers.

Let P be the assertion that d is even. Let W be the assertion that exactly one among a, b, c are even, and let T be the assertion that all three are even.

Prove by cases that

$$P \text{ IFF } [W \text{ OR } T].$$

(b) Now suppose that

$$w^2 + x^2 + y^2 = z^2,$$

where w, x, y, z are nonnegative integers. Let P be the assertion that z is even, and let R be the assertion that all three of w, x, y are even. Prove by cases that

$$P \text{ IFF } R.$$

Hint: An odd number equals $2m + 1$ for some integer m, so its square equals $4(m^2 + m) + 1$.

Problem 3.
Sloppy Sam is trying to prove a certain proposition P. He defines two related propositions Q and R, and then proceeds to prove three implications:

$$P \text{ IMPLIES } Q, \qquad Q \text{ IMPLIES } R, \qquad R \text{ IMPLIES } P.$$

He then reasons as follows:

> If Q is true, then since I proved (Q IMPLIES R), I can conclude that R is true. Now, since I proved (R IMPLIES P), I can conclude that P is true. Similarly, if R is true, then P is true and so Q is true. Likewise, if P is true, then so are Q and R. So any way you look at it, all three of P, Q and R are true.

(a) Exhibit truth tables for

$$(P \text{ IMPLIES } Q) \text{ AND } (Q \text{ IMPLIES } R) \text{ AND } (R \text{ IMPLIES } P) \qquad (*)$$

and for

$$P \text{ AND } Q \text{ AND } R. \qquad (**)$$

Use these tables to find a truth assignment for P, Q, R so that (*) is **T** and (**) is **F**.

(b) You show these truth tables to Sloppy Sam and he says "OK, I'm wrong that P, Q and R all have to be true, but I still don't see the mistake in my reasoning. Can you help me understand my mistake?" How would you explain to Sammy where the flaw lies in his reasoning?