Mathematics for Computer Science
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The Logic of Propositions

Proving Validity

Instead of truth tables, can try to prove valid formulas symbolically using axioms and deduction rules

A Proof System

Another approach is to start with some valid formulas (axioms) and deduce more valid formulas using proof rules

A Proof System

Lukasiewicz' proof system is a particularly elegant example of this idea.

A Proof System

Lukasiewicz' proof system is a particularly elegant example of this idea. It covers formulas whose only logical operators are IMPLIES ($\rightarrow$) and NOT.

Lukasiewicz' Proof System

Axioms:

1) $(\neg P \rightarrow P) \rightarrow P$
2) $P \rightarrow (\neg P \rightarrow Q)$
3) $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

The only rule: modus ponens

Lukasiewicz' Proof System

Three Axioms:

$(\overline{P} \rightarrow P) \rightarrow P$

$P \rightarrow (\overline{P} \rightarrow Q)$

$(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

One Rule: modus ponens

Lukasiewicz' Proof System

Prove formulas by starting with axioms and repeatedly applying the inference rule.

To illustrate the proof system we'll do an example, which you may safely skip.

Lukasiewicz' Proof System

Prove formulas by starting with axioms and repeatedly applying the inference rule.

For example, to prove:

$$P \longrightarrow P$$

A Lukasiewicz' Proof

3rd axiom:

$$(P \longrightarrow Q) \longrightarrow$$
$$((Q \longrightarrow R) \longrightarrow (P \longrightarrow R))$$

replace R by P

A Lukasiewicz' Proof

3rd axiom:

$$(P \longrightarrow Q) \longrightarrow$$
$$((Q \longrightarrow P) \longrightarrow (P \longrightarrow P))$$

replace Q by $(\overline{P} \longrightarrow P)$

A Lukasiewicz' Proof

3rd axiom:

Axiom 2)

$$(P \longrightarrow (\overline{P} \longrightarrow P)) \longrightarrow$$
$$(((\overline{P} \longrightarrow P) \longrightarrow P) \longrightarrow (P \longrightarrow P))$$

A Lukasiewicz' Proof

so apply modus ponens:

$$\overbrace{(P \to (\overline{P} \to P))}^{\text{Axiom 2)}} \to$$
$$(((\overline{P} \to P) \to P) \to (P \to P))$$

A Lukasiewicz' Proof

so apply modus ponens:

$$\overbrace{((\overline{P} \to P) \to P)}^{\text{Axiom 1)}} \to (P \to P))$$

A Lukasiewicz' Proof

so apply modus ponens:

$$(P \to P)$$

QED

Lukasiewicz is Sound

The 3 Axioms are all valid (verify by truth table). We know modus ponens is sound. So every provable formula is also valid.

Lukasiewicz is Complete

Conversely, every valid (NOT,$\rightarrow$)-formula is provable:

system is "complete"

Not hard to verify but would take a full lecture; we omit it.

Validity/SAT still difficult!

Deduction systems in general no better than truth tables.

No efficient method for verifying validity is known.

