Validity & Soundness

Propositional Validity

True for all truth-values.
Example:
(P IMPLIES Q) OR (Q IMPLIES P)

Predicate Calculus Validity

True for all domains and predicates. Example:
∀z.[P(z) AND Q(z)] IMPLIES [∀x.P(x) AND ∀y.Q(y)]

Predicate Calculus Validity

True for all domains and predicates*. Example:
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*aka tautology
DeMorgan's Law for Quantifiers

Another valid formula:

\[
\neg(\forall x. P(x)) \iff \exists y. \neg(P(y))
\]

Another valid formula:

\[
\neg(\text{AND}_x P(x)) \iff \text{OR}_y \neg(P(y))
\]

Proving Validity

\[
\forall z. [P(z) \land Q(z)] \implies [\forall x. P(x) \land \forall y. Q(y)]
\]

Proof strategy: assume left side is \(T\), then prove right side is \(T\)

Proof of Validity

\[
\forall z. [Q(z) \land P(z)] \rightarrow [\forall x. Q(x) \land \forall y. P(y)]
\]

Proof: Assume left hand side. That is, for all values of \(z\) in the domain, \(Q(z) \land P(z)\) is true.

Suppose \(\text{val}(z) = c\), an element in the domain. Then \(Q(c) \land P(c)\) holds, and so \(Q(c)\) by itself holds. But \(c\) could have been any element of the domain.

So we conclude \(\forall x. Q(x)\). \((\text{by UG})\)

Similarly conclude \(\forall y. P(y)\). Therefore, \(\forall x. Q(x) \land \forall y. P(y)\) \(\text{QED}\)
Similar Example is Not Valid

\[ \forall z. [P(z) \lor Q(z)] \implies [\forall x. P(x) \lor \forall y. Q(y)] \]

Proof: Give counter-model, where left side of IMPLIES is \( T \), but right side is \( F \).
Namely, let domain ::= \{1, 2\}, \( Q(z) ::= [z = 1], \ P(z) ::= [z = 2] \).

Universal Generalization (UG)

\[
\begin{align*}
F(c) \\
\forall x. F(x)
\end{align*}
\]

where \( c \) is a constant symbol that has not appeared earlier.

Universal Generalization (UG)

\[
\begin{align*}
F(c) \\
\forall x. F(x)
\end{align*}
\]

\( c \) is a "fresh symbol".

Subtlety:

\[
F(c) \text{ does not imply } \forall x. F(x)
\]
Universal Generalization (UG)

\[
\frac{F(c)}{\forall x.F(x)}
\]

...unlike propositional case instead have weaker notion of Soundness:

Weaker notion of Soundness

If \( F(c) \) is valid then \( \forall x. F(x) \) is valid