Mathematics for Computer Science
MIT 6.042J/18.062J

Euler's Function

Euler ϕ function

$\phi(n) ::= \#\ k \in [0,n)$
s.t. k rel. prime to n

Euler ϕ function

$\phi(n) ::= \#\ k \in [0,n)$
s.t. $\gcd(k,n) = 1$

Euler ϕ function

$\text{gcd1}\{n\} ::=$
$\{k \in [0,n) \mid \gcd(k,n)=1\}$

so $\phi(n) = |\text{gcd1}\{n\}|$

(some books write n^* for gcd1{n})

Euler ϕ function

gcd1{n} ::=

$\{k \in [0,n) \mid \gcd(k,n)=1\}$

gcd1{7} = {1,2,3,4,5,6}

gcd1{12} =

{0,1,2,3,4,5,6,7,8,9,10,11}

Euler ϕ function

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Euler ϕ function

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$\phi(7) = 6$

$\phi(12) =$

$|\{1, 5, 7, 11\}|$

Euler ϕ function

$\text{gcd1}\{n\} ::=$

$\{k \in [0,n) \mid \gcd(k,n)=1\}$

$$\phi(7) = 6$$

$$\phi(12) = 4$$

Calculating ϕ

If p prime, everything in $[1,p)$ is rel. prime to p, so

$$\boxed{\phi(p) = p - 1}$$

Calculating ϕ

$\phi(9)$? $0,1,2,3,4,5,6,7,8$

k rel. prime to 9 iff

k rel. prime to 3

3 divides every 3rd number

so, $\phi(9) = 9-(9/3) = 6$

Calculating $\phi(p^k)$

$0,1,\ldots,p,\ldots,2p,\ldots\ldots,p^k-p,\ldots,p^k-1$

p divides every pth number

p^k/p of these numbers are not rel. prime to p^k

Calculating $\phi(p^k)$

so

$$\phi(p^k) = p^k - p^k/p$$

Calculating $\phi(p^k)$

so

$$\boxed{\phi(p^k) = p^k - p^{k-1}}$$

Calculating $\phi(a \cdot b)$

Lemma:

For a, b relatively prime,

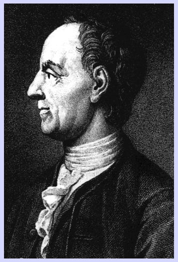
$$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

pf: Given as problem.
Also later by "counting."

Calculating $\phi(a \cdot b)$

$$\phi(12) = \phi(3 \cdot 4)$$
$$= \phi(3) \cdot \phi(4)$$
$$= (3 - 1) \cdot (2^2 - 2^{2-1})$$
$$= 2 \cdot (4 - 2) = 4$$

Euler's Theorem



For k relatively prime to n,

$$k^{\phi(n)} \equiv 1 \pmod{n}$$