Midterm Exam April 12

Your name:____

	1PM	**32-082 Table (1–13):**	_______
Identify your Team:	**1PM**	**32-044 Table (A–K):**	_______
	2:30PM	**32-044 Table (A–K):**	_______

- This exam is **closed book** except for a 2-sided cribsheet. Total time is 90 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text (unless explicitly instructed otherwise).
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	20		
2	20		
3	20		
4	20		
5	20		
Total	100		

Problem 1 (Diagonalization) (20 points).
As we did in PSET 4 Problem #2, let $\mathbb{N}^\omega$ be the set of infinite sequences of natural numbers, and say that a sequence $(a_0, a_1, a_2, \ldots) \in \mathbb{N}^\omega$ is *strictly increasing* if $a_0 < a_1 < a_2 < \cdots$. Define the subset Inc $\subset \mathbb{N}^\omega$ to be the set of strictly increasing sequences.

Carefully prove that Inc is uncountable by a direct **diagonalization argument over Inc**. Your proof should be self-contained, i.e., written in a way that doesn't assume the reader is familiar with similar diagonalization proofs.

Note: PSET 4 Problem #2 provides a different way to prove that Inc is uncountable, by showing Inc bij $\mathbb{N}^\omega$. Please do *not* cite or duplicate that bijection here, as it isn't helpful in constructing a proof that diagonalizes Inc.

Problem 2 (**Bipartite Matching**) (**20 points**). **(a)** Prove that the bipartite graph G in Figure 1 has no perfect matching by exhibiting a bottleneck in it. Briefly explain why your choice is a bottleneck.

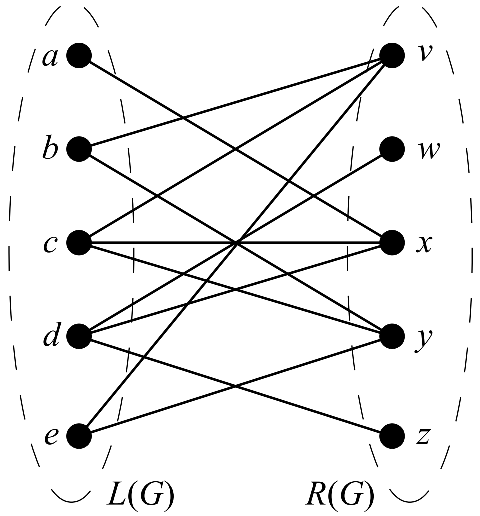

Figure 1 Bipartite graph G.

(b) The bipartite graph H in Figure 2 has an easily verified property that implies it has a matching that covers $L(H)$ (without needing to actually find a matching). What is the property and why does H have it?

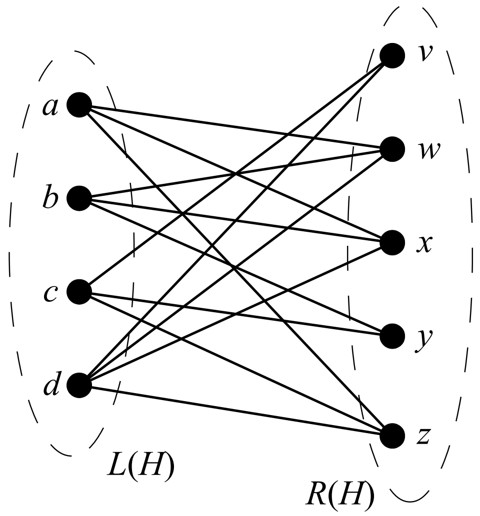

Figure 2 Bipartite Graph H.

(c) Find a matching that covers $L(H)$ and includes edge $\langle d\text{—}x \rangle$.

Problem 3 (Partial Orders) (20 points).

A tennis tournament[1] among a set of players consists of a series of two-player matches. Usually the objective is to determine a single best player. The organizers of the Math for Computer Science tournament want to do more: they want to find a linear ranking of all the players. To avoid controversy, they want to avoid the awkward situation of having a sequence of players each of whom beats the next player in the sequence and then having last player beat the first. So the organizers will keep a running record of who beat whom during the tournament, and they never allow simultaneous matches whose outcomes could lead to an awkward situation.

Knowledge of binary relations can help the organizers in arranging the tournament. Namely, at any stage of the tournament, the organizers have a record of who lost to whom. Mathematically, we can say that there is a binary relation L on players where $p \; L \; q$ means that player p lost a match to player q. No awkward situations means that the positive length walk relation L^+ is a strict partial order. Indicate which of the following partial order concepts correspond to the properties (a)–(i) of the partial order L^+. (Part (j) asks for a numerical answer, not a fill-in-the-blank.)

Partial Order Concepts

comparable, incomparable, maximum, maximal, minimum, minimal,
a chain, an antichain, reflexive, irreflexive, asymmetric,
a topological sort, a linear order.

(a) An unbeaten player so far is a _____________ element.

(b) A player who has lost every match he was in is a _____________ element.

(c) A player who is sure to rank first at the end of the tournament is a _____________ element.

(d) A set of players whose rankings relative to each other are unique is _____________.

(e) Two players can be matched in the next stage of the tournament only if they are _____________ elements.

(f) The final ranking at the end of the tournament will be _____________.

(g) No more matches are possible if and only if L^+ is ____.

(h) A set of players any two of whom could be paired up to play the next match is _____________.

(i) The fact that no player loses to himself corresponds to L^+ being _____________.

(j) If there are 256 players, what is the smallest number of matches that could possibly have been played in a completed tournament? _______

[1] Note: We are *not* talking about "tournament digraphs" like the ones defined in the King Chicken problem. In this problem, the word "tournament" simply means "competition."

Problem 4 (**Trees, Graph Coloring**) (**20 points**). **(a)** Prove by induction that every tree is 2-colorable. As usual, be sure to carefully specify your induction hypothesis.

(b) From part (a) it follows that *almost* all trees T have chromatic number $\chi(T) = 2$. What are the exceptions, and what are their chromatic numbers?

Hint: Think small. Recall that a simple graph, by definition, has a nonempty set of vertices.

Problem 5 (Graph Isomorphisms) (20 points).
List *all* of the isomorphisms between the two graphs in Figure 3. Explain why there are no others.

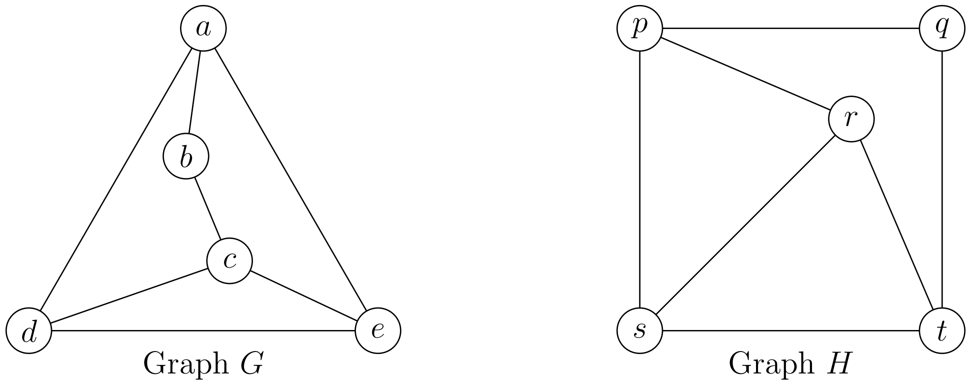

Figure 3 Graphs with several isomorphisms