Midterm Exam February 22

Your name:____

	1PM	**32-082 Table (1–13):**	_______
Indicate your Identify your Team:	**1PM**	**32-044 Table (A–K):**	_______
	2:30PM	**32-044 Table (A–K):**	_______

- This exam is **closed book** except for a 2-sided cribsheet. Total time is 90 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	0		
2	0		
3	0		
4	0		
5	0		
Total	0		

Problem 1 ().
Use the Well Ordering Principle to prove that

$$n \leq 3^{n/3} \tag{*}$$

for every nonnegative integer n.
Hint: Verify (*) for $n \leq 4$ by explicit calculation.

Problem 2 ().
Let $f : D \to D$ be a total function from some nonempty set D to itself. In the following propositions, x and y are variables ranging over D, and g is a variable ranging over total functions from D to D. Circle all of the propositions that are equivalent to the proposition that f is an *injection*. No explanations are necessary.

(i) $\forall x \forall y.\ x = y \text{ OR } f(x) \neq f(y)$

(ii) $\forall x \forall y.\ x = y \text{ IMPLIES } f(x) = f(y)$

(iii) $\forall x \forall y.\ x \neq y \text{ IMPLIES } f(x) \neq f(y)$

(iv) $\forall x \forall y.\ f(x) = f(y) \text{ IMPLIES } x = y$

(v) $\text{NOT}[\exists x \exists y.\ x \neq y \text{ AND } f(x) = f(y)]$

(vi) $\text{NOT}[\exists y \forall x.\ f(x) \neq y]$

(vii) $\exists g \forall x.\ g(f(x)) = x$

(viii) $\exists g \forall x. f(g(x)) = x$

Problem 3 ().
There is a bucket containing more blue balls than red balls. As long as there are more blues than reds, any one of the following rules may be applied to add and/or remove balls from the bucket:

(i) Add a red ball.

(ii) Remove a blue ball.

(iii) Add two reds and one blue.

(iv) Remove two blues and one red.

(a) Starting with 10 reds and 16 blues, what is the largest number of balls the bucket will contain by applying these rules? ________

Let b be the number of blue balls and r be the number of red balls in the bucket at any given time.

(b) Prove that $b - r \geq 0$ is a preserved invariant of the process of adding and removing balls according to rules (i)–(iv).

(c) Find a nonnegative integer-valued derived variable that is strictly decreasing, and use it to prove that no matter how many balls the bucket contains, repeatedly applying rules (i)–(iv) will eventually lead to a state where no further rule can be applied.

Problem 4 ().
Four unfortunate children want to be adopted by four foster families of ill repute. A child can only be adopted by one family, and a family can only adopt one child. Here are their preference rankings (most-favored to least-favored):

Child	Families
Bottlecap:	Hatfields, McCoys, Grinches, Scrooges
Lucy:	Grinches, Scrooges, McCoys, Hatfields
Dingdong:	Hatfields, Scrooges, Grinches, McCoys
Zippy:	McCoys, Grinches, Scrooges, Hatfields

Family	Children
Grinches:	Zippy, Dingdong, Bottlecap, Lucy
Hatfields:	Zippy, Bottlecap, Dingdong, Lucy
Scrooges:	Bottlecap, Lucy, Dingdong, Zippy
McCoys:	Lucy, Zippy, Bottlecap, Dingdong

(a) Exhibit two different stable matching of Children and Families.

Family	Child in 1st match	Child in 2nd match
Grinches:		
Hatfields:		
Scrooges:		
McCoys:		

(b) Examine the matchings from part (a), and explain why these matchings are the only two possible stable matchings between Children and Families.

Hint: In general, there may be many more than two stable matchings for the same set of preferences.

Problem 5 ().
The set RecMatch of strings of matched brackets is defined recursively by

Base case: The empty string $\lambda \in$ RecMatch.

Constructor case: If $s, t \in$ RecMatch, then

$$[\, s \,]\, t \in \text{RecMatch}.$$

An alternative definition is the set AmbRecMatch defined recursively by

Base case: $\lambda \in$ AmbRecMatch.

Constructor cases: If $s, t \in$ AmbRecMatch, then

- [s] $\in$ AmbRecMatch, and
- $st \in$ AmbRecMatch.

Fill in the following parts of a proof by structural induction that

$$\text{RecMatch} \subseteq \text{AmbRecMatch}. \qquad (*)$$

(As a matter of fact, AmbRecMatch = RecMatch, though we won't prove this here.)

(a) State an **induction hypothesis** suitable for proving (*) by structural induction.

(b) State and prove the **base case**.

(c) Prove the **inductive step**.

An advantage of the RecMatch definition is that it is *unambiguous*, while the definition of AmbRecMatch is ambiguous.

(d) Give an example of a string whose membership in AmbRecMatch is ambiguously derived.

(e) Briefly explain what advantage unambiguous recursive definitions have over ambiguous ones. (Remember that "ambiguous recursive definition" has a technical mathematical meaning which does not imply that the ambiguous definition is unclear.)