A Magic Trick

Audience chooses 5 cards
Assistant reveals 4 of them
Magician announces 5th card!

Magic Trick Revealed (I)

Among 5 cards chosen:
  at least 2 have the same suit
(Pigeonhole Principle)

A hides one, lists the other one 1st

Aha! The first card has the same suit as the hidden card!

Magic Trick Revealed (II)

How does M figure out the rank of the hidden card?

Aha! Look at the order of the other 3 cards!
Magic Trick Revealed (II)

Fix ordering of the deck:

A♣ < A♦ < A♥ < A♠ < 2♣ < 2♦ < 2♥ < 2♠ < ...

K♣ < K♦ < K♥ < K♠

Magic Trick Revealed (II)

Possible orders for the remaining 3 cards:

{ SML, SLM, MSL, MLS, LSM, LMS }

Magic Trick Revealed (II)

Wait! Only have 6 sequences of the remaining 3 cards, but 12 possible hidden cards of the known suit!

Of two cards with the same suit, choosing which to reveal can give 1 more bit of information!

Aha!

Clockwise Distance

The smaller clockwise distance between 2 card ranks is at most 6:

Reveal the other card

Hide card with smaller offset.
Magic Trick Revealed (Finally)

- The first card determines the hidden suit (♦ ♠ ♥ ♣).
- Hidden rank (A … K) = first-card rank + offset ($\leq 6$).
- Offset given by order of remaining 3 cards:
  \[ SML = 1, \ SLM = 2, \ MSL = 3, \ MLS = 4, \ LSM = 5, \ LMS = 6. \]

Example

Hidden: 
First: 
Offset = 1 = SML:

won’t work with 4-card hands

A can reveal
\[
\binom{52}{4} = 270,725
\]
possible 4-card hands
\[
\binom{52}{3} = 132,600
\]
possible 3-card lists

so at least
\[
\left[ \frac{270,725}{132,600} \right] = 3
\]

hands map to the same list
- M can’t tell which!