Hall's Theorem

A match is a total injective function
\[ m : L(H) \rightarrow R(H) \]
that follows edges

A match is a total injective function
\[ m : L(H) \rightarrow R(H) \]
\[ l \rightarrow m(l) \in E(H) \]
Bipartite match

A match is a total injective function
\[ m: L(H) \rightarrow R(H) \]

graph(m) ⊆ E(H)

Hall’s Theorem

Hall’s condition

If \(|S| \leq |E(S)|\) for all sets \(S \subseteq L(H)\), then there is a match.

How to verify no bottlenecks?

fairly efficient matching procedure is known (explained in algorithms subjects)
...but there is a special situation that ensures a match...

How to verify no bottlenecks?

If every girl likes \(\geq d\) boys, and every boy likes \(\leq d\) girls, then no bottlenecks.

a degree-constrained bipartite graph
Degree constrained implies Hall condition
If every girl likes $\geq d$ boys, and every boy likes $\leq d$ girls, then no bottlenecks.
proof:

Hall.9

proof:
say set $S$ of girls has $e$ incident edges:
$$d \cdot |S| \leq e \leq d \cdot |E(S)|$$
so $|S| \leq |E(S)|$
no bottleneck QED

Proof of Hall's Theorem
Suppose no bottlenecks.

Lemma: No bottlenecks within any set $S$ of girls.
obviously

Hall.11

Lemma: If $S$ a set of girls with $|S| = |E(S)|$, then no bottlenecks between $S$ and $E(S)$ either

Hall.12
bottleneck between $\tilde{S}$ & $E(S)$?

Then $T \cup S$ is a bottleneck.
Proof of Hall’s Theorem

No bottlenecks implies there is a perfect match.

proof:
by strong induction on # girls

Case 1: there is a nonempty proper subset $S$ of girls with $|S| = |E(S)|$.
by Lemmas, no bottlenecks in Hall graph $(S, E(S))$, and none in $(S, E(S))$

by induction, match $(S, E(S))$ and $(S, E(S))$ separately.

by induction, match $(S, E(S))$ and $(S, E(S))$ separately. Matchings don’t overlap, so union is a complete matching.
Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets $S$. Pick a girl, $g$.

Match $g$ with $b$. Removing $b$ still leaves $|S| \leq |E(S)|$, so no bottlenecks.
Hall’s Theorem

Case 2: \(|S| < |E(S)|\) for all nonempty proper subsets \(S\). By induction, can match remaining girls & boys. This match along with \(g—b\) is complete match. QED