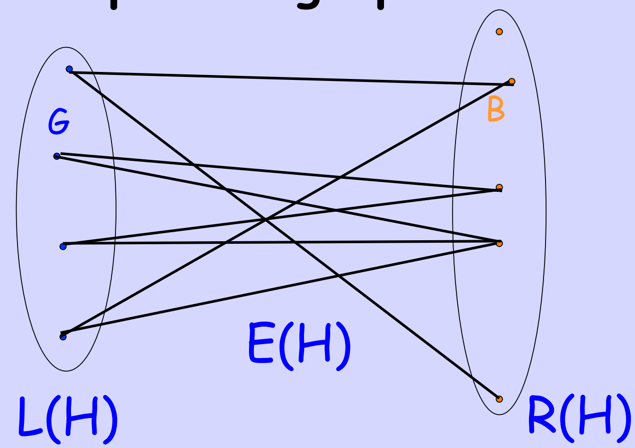
Mathematics for Computer Science
MIT 6.042J/18.062J

Hall's Theorem

Bipartite graph H



Bipartite match

A match is a

total injective function

$$m:L(H)\to R(H)$$

that follows edges

Bipartite match

A match is a

total injective function

$$m:L(H)\to R(H)$$

$$l - m(l) \in E(H)$$

Bipartite match

A match is a

total injective function

$$m:L(H)\to R(H)$$

$$\text{graph}(m) \subseteq E(H)$$

Hall's Theorem

Hall's condition

If $|S| \leq |E(S)|$ for all sets $S \subseteq L(H)$

then there is a match.

How to verify no bottlenecks?

fairly efficient matching procedure is known

(explained in algorithms subjects)

…but there is a special situation that ensures a match…

How to verify no bottlenecks?

If every girl likes $\geq$ d boys,
and every boy likes $\leq$ d girls,

then no bottlenecks.

a degree-constrained bipartite graph

Degree constrained implies Hall condition

If every girl likes $\geq$ d boys,
and every boy likes $\leq$ d girls,
then no bottlenecks.

proof:

Degree constrained implies Hall condition

If every girl likes $\geq$ d boys,
and every boy likes $\leq$ d girls,
then no bottlenecks.

proof: say set S of girls has e incident edges:

$$d \cdot |S| \leq e \leq d \cdot |E(S)|$$

$$\text{so } |S| \leq |E(S)|$$

no bottleneck QED

Proof of Hall's Theorem

Suppose no bottlenecks.

Lemma: No bottlenecks within any set S of girls.

obviously

Proof of Hall's Theorem

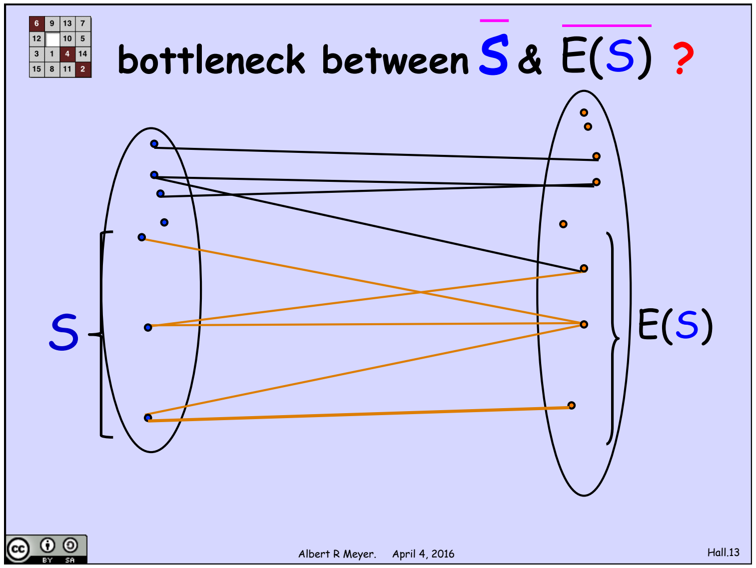
Suppose no bottlenecks.

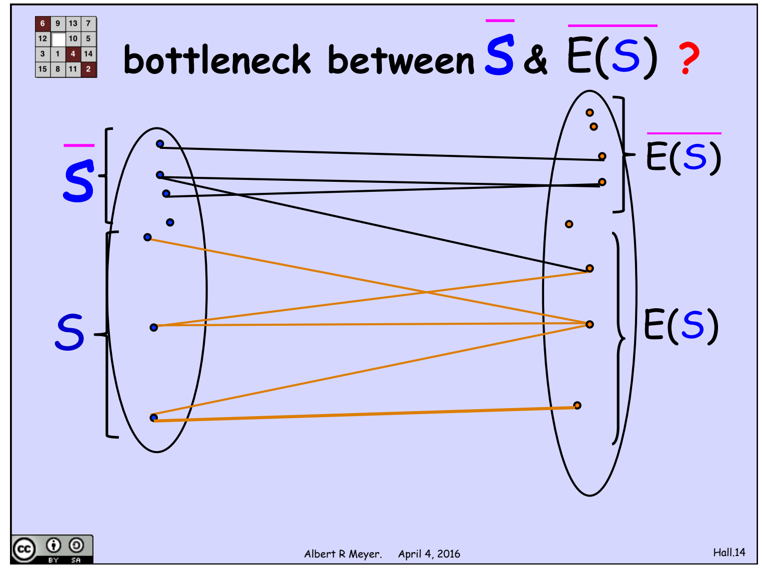
Lemma: If S a set of girls with

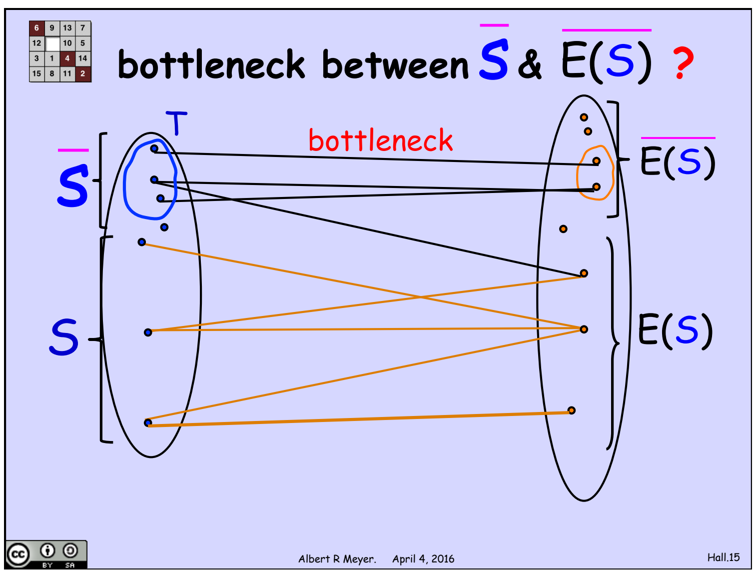
$$|S| = |E(S)|,$$

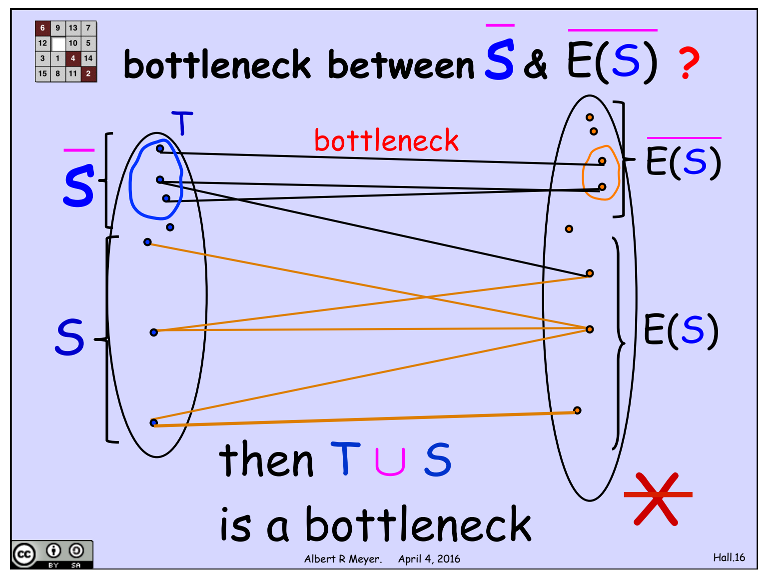
then no bottlenecks between

$\overline{S}$ and $\overline{E(S)}$ either


bottleneck between $\overline{S}$ & $\overline{E(S)}$?
S
E(S)
Albert R Meyer. April 4, 2016
Hall.13


bottleneck between $\overline{S}$ & $\overline{E(S)}$?
$\overline{S}$
$\overline{E(S)}$
S
E(S)
Albert R Meyer. April 4, 2016
Hall.14


bottleneck between $\overline{S}$ & $\overline{E(S)}$?
T
bottleneck
$\overline{S}$
$\overline{E(S)}$
S
E(S)
Albert R Meyer. April 4, 2016
Hall.15


bottleneck between $\overline{S}$ & $\overline{E(S)}$?
T
bottleneck
$\overline{S}$
$\overline{E(S)}$
S
E(S)
then T ∪ S
is a bottleneck
Albert R Meyer. April 4, 2016
Hall.16

Proof of Hall's Theorem

No bottlenecks implies there is a perfect match.

proof:

by strong induction on # girls

Proof of Hall's Theorem

Case 1: there is a nonempty proper subset S of girls with

$$|S|=|E(S)|.$$

by Lemmas, no bottlenecks in Hall graph $(S, E(S))$,

and none in $\left(\overline{S}, \overline{E(S)}\right)$

Proof of Hall's Theorem

by induction, match $(S, E(S))$ and $\left(\overline{S}, \overline{E(S)}\right)$ separately.

Proof of Hall's Theorem

by induction, match $(S, E(S))$ and $\left(\overline{S}, \overline{E(S)}\right)$ separately. Matchings don't overlap, so union is a complete matching.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

Pick a girl, g.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

Pick a girl, g. She must be compatible with some boy, b.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

Match g with b.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

Match g with b. Removing b still leaves $|S| \leq |E(S)|$, so no bottlenecks.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

By induction, can match remaining girls & boys. This match along with g—b is complete match. QED