Mathematics for Computer Science
MIT 6.042J/18.062J

Min-Gray Edges give Min-Weight Tree

Connected Graph G

Let G be a connected, weighted simple graph.
Assume all edges have different weights.

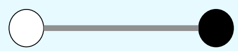
Black-white coloring

Let G be a connected, weighted simple graph.
Color each vertex of G **black** or white

Black-white coloring

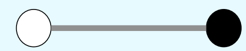
Let G be a connected, weighted simple graph.
Color each vertex of G **black** or white, not all same color.

Gray Edges

A gray edge connects vertices with different colors: 

There must be a gray edge since G connected

Gray Edges

A gray edge connects vertices with different colors: 

Let e be a min-weight gray edge.

Min Gray Edge Necessary

Theorem:

e is an edge of every min-weight spanning tree (MST)

Min Gray Edges Sufficient

There is a spanning tree built of min-weight gray edges

-- from previous slides.

Unique MST

Corollary: There is a unique MST.

min-weight gray = MST

Corollary: There is a unique MST. It consists of all min-weight gray edges under black-white colorings.

Gray Edge Swap Lemma

Let C be a connected spanning subgraph (css) of G.

Gray Edge Swap Lemma

Suppose e not an edge of css C.

Gray Edge Swap Lemma

Suppose e not an edge of css C. Then there is an edge g of C:

(i) $wt(e) < wt(g)$

(ii) $C-g+e$ is a css

Gray Edge Swap Lemma

So C is not minimum css because $C-g+e$ has smaller weight.

Min Gray Edge Necessary

So e is necessarily in any min-weight css.

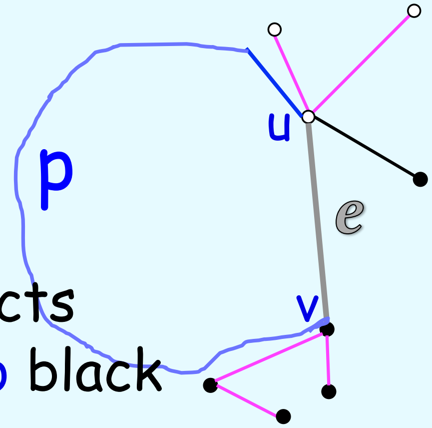
Proof of Swap Lemma

Say $e = \langle u - v \rangle$.

C connected, so have path

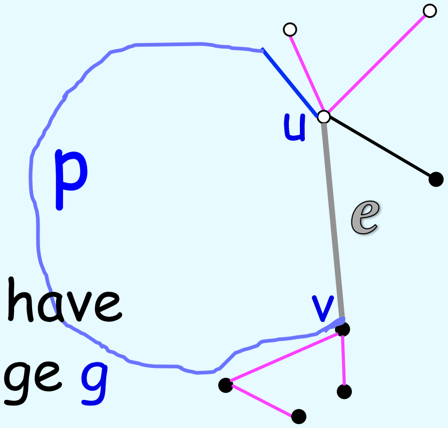
$$\vec{p} = u\text{-}\cdots\text{-}v.$$

Proof of Swap Lemma



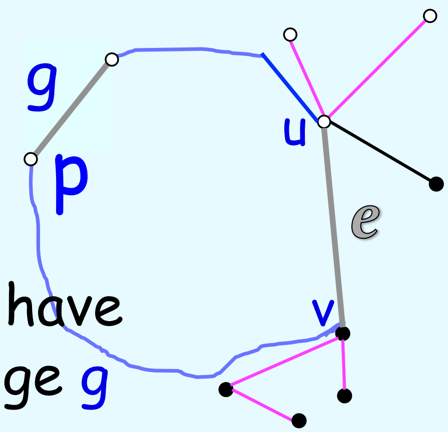
p connects white to black

Proof of Swap Lemma



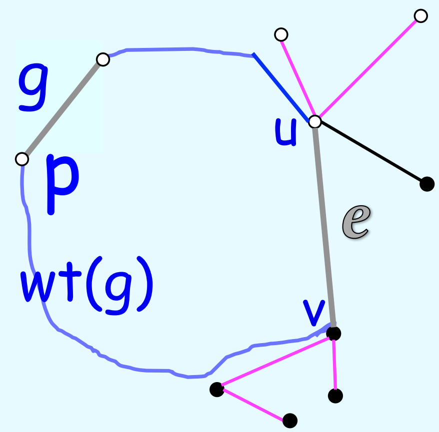
p must have gray edge g

Proof of Swap Lemma



p must have gray edge g

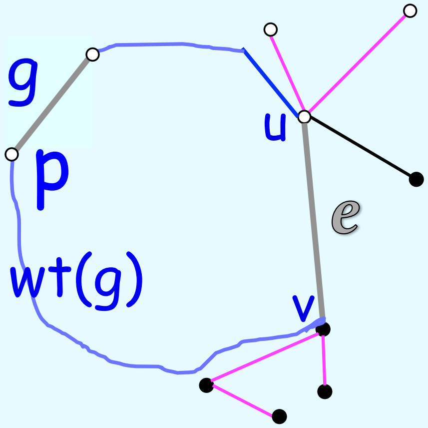
Proof of Swap Lemma



$wt(e) < wt(g)$

Proof of Swap Lemma

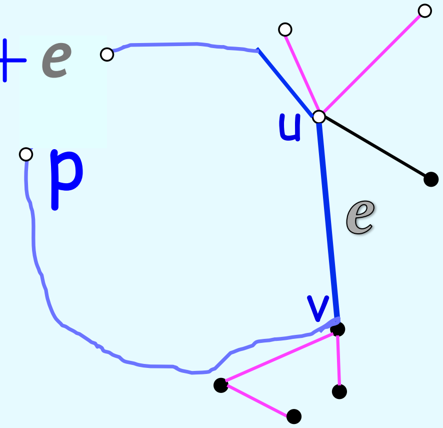
$C - g$



$\mathrm{wt}(e) < \mathrm{wt}(g)$

Proof of Swap Lemma

$C - g + e$

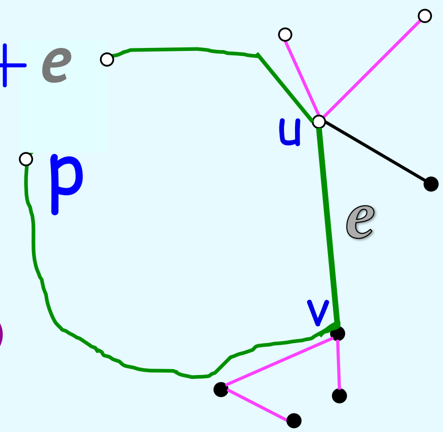


Proof of Swap Lemma

$C - g + e$ is connected:
end-points of g connected
by path

Proof of Swap Lemma

$C - g + e$



QED