Min-Gray Edges give Min-Weight Tree

Let $G$ be a connected, weighted simple graph. Assume all edges have different weights.

Black-white coloring
Let $G$ be a connected, weighted simple graph. Color each vertex of $G$ black or white, not all same color.
A gray edge connects vertices with different colors:  

There must be a gray edge since $G$ connected

Let $e$ be a min-weight gray edge.

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**Min Gray Edge Necessary**

Theorem: $e$ is an edge of every min-weight spanning tree (MST)

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**Min Gray Edges Sufficient**

There is a spanning tree built of min-weight gray edges

-- from previous slides.
Corollary: There is a unique MST. It consists of all min-weight gray edges under black-white colorings.

Gray Edge Swap Lemma

Let $C$ be a connected spanning subgraph (CSS) of $G$. If $e$ is not an edge of CSS $C$, then...
Gray Edge Swap Lemma

Suppose $e$ not an edge of css $C$. Then there is an edge $g$ of $C$:

(i) $\text{wt}(e) < \text{wt}(g)$
(ii) $C - g + e$ is a css

Min Gray Edge Necessary

So $e$ is necessarily in any min-weight css.

Proof of Swap Lemma

Say $e = <u - v>$. $C$ connected, so have path $\vec{p} = u \cdots v$. 
Proof of Swap Lemma

\( p \) connects \( u \) white to black \( v \)

Proof of Swap Lemma

\( p \) must have gray edge \( g \)

Proof of Swap Lemma

\( p \) must have gray edge \( g \)

Proof of Swap Lemma

\( wt(e) < wt(g) \)
Proof of Swap Lemma

\[ C - g \]

\[ g \]

\[ wt(e) < wt(g) \]

Proof of Swap Lemma

\[ C - g + e \]

is connected:
end-points of \( g \) connected by path

QED