Mathematics for Computer Science

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Gambler's Ruin Expected Time

How Long Till the End?

How many bets expected till Gambler either hits target or get ruined?

Expected number of bets

$$e_n ::= Ex[\#\text{bets with \$}n\text{ start}]$$

$$Ex[e_n \mid \text{win 1st bet}] = 1 + e_{n+1}$$

$$Ex[e_n \mid \text{lose 1st bet}] = 1 + e_{n-1}$$

Apply Total Expectation

$$e_n =$$
$$Ex[e_n \mid \text{win 1st bet}] \cdot Pr[\text{win 1st bet}] +$$
$$Ex[e_n \mid \text{lose 1st bet}] \cdot Pr[\text{lose 1st bet}]$$
$$= \left(1 + e_{n+1}\right)p + \left(1 + e_{n-1}\right)q$$

Linear recurrence

$$e_{n+1} = \frac{1}{p} e_n - \frac{q}{p} e_{n-1} - \frac{1}{p}$$

Expected number of bets

Solve linear recurrence as usual.
Elegant result in the fair case:

$$e_n = n(T-n)$$

$$= (\text{initial stake}) \cdot (\text{intended profit})$$

Expected number of fair bets

For example, starting with $1 aiming to reach $1000, expect to make 999 bets (and most likely go broke)
Problem: There must be an intuitive proof. Find one.

Keep Playing until Ruin

Suppose keep playing until ruined? …that is, Target = ∞.

In unfavorable game, ruin is certain:

$$\Pr[\text{ruin}] \geq 1 - \left(\frac{1}{r}\right)^{\text{intended profit}}$$

Keep Playing until Ruin

Suppose keep playing until ruined? …that is, Target $= \infty$.

In unfavorable game,

ruin is certain:

$$\Pr[\text{ruin}] \geq 1 - \left(\frac{p}{q}\right)^{\infty} = 1$$

Keep Playing until Ruin

Suppose keep playing until ruined? …that is, Target $= \infty$.

In unfavorable game, and

$$E[\#\text{ bets from } \$n] = \Theta(n)$$

Keep Playing until Ruin

Suppose keep playing until ruined? …that is, Target $= \infty$.

In fair game,

ruin is also certain, but

expected time to ruin $= \infty$

(class problem)

Keep Playing until Ruin

Suppose keep playing until ruined? …that is, Target $= \infty$.

In favorable game,

ruin is not certain:

play forever with prob > 0