Massachusetts Institute of Technology
6.042J/18.062J, Spring '18: Mathematics for Computer Science
Dr. Zachary Abel

May 23
revised Thursday 24th May, 2018, 10:17

Final Examination, Unit 5

Your name:______________________________

- This exam is **closed book** except for two 2-sided cribsheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	20		
2	20		
3	20		
Total	60		

Problem 1 (Probability) (20 points).
Suppose that *Let's Make a Deal* is played according to slightly different rules and with a red goat and a blue goat. There are three doors, with a prize hidden behind one of them and one goat behind each of the others. No doors are opened until the contestant makes a final choice to stick or switch. The contestant is allowed to pick a door and ask a certain question that the host then answers honestly. The contestant may then stick with their chosen door, or switch to either of the other doors.

(a) If the contestant asks "is there is a goat behind one of the unchosen doors?" and the host answers "yes," is the contestant more likely to win the prize if they stick, switch, or does it not matter? Clearly identify the probability space of outcomes and their probabilities you use to model this situation. What is the contestant's probability of winning if he uses the best strategy?

(b) If the contestant asks "is the *red* goat behind one of the unchosen doors?" and the host answers "yes," is the contestant more likely to win the prize if they stick, switch, or does it not matter? Clearly identify the probability space of outcomes and their probabilities you use to model this situation. What is the contestant's probability of winning if he uses the best strategy?

Problem 2 (**Markov and Chebyshev Bounds**) (**20 points**).

You have a biased coin which flips Heads with probability p. You flip the coin n times. The coin flips are all mutually independent. Let H be the number of Heads.

(a) Write a closed-form (no summations) expression in terms of p and n for $\text{Ex}[H]$, the expected number of Heads. Briefly explain your answer.

$\boxed{}$

(b) Write a closed-form expression in terms of p and n for $\text{Var}[H]$, the variance of the number of Heads. Briefly explain your answer.

$\boxed{}$

(c) Write a closed-form expression in terms of p for the upper bound that Markov's Theorem gives for the probability that the number of Heads is larger than the expected number by at least 1% of the number of flips, that is, by $n/100$.

$\boxed{}$

(d) Show that the upper bound given by Chebyshev's Theorem for the probability that H differs from $\text{Ex}[H]$ by at least $n/100$ is

$$100^2 \frac{p(1-p)}{n}.$$

Problem 3 (Inclusion-Exclusion) (20 points).
A permutation of the letters a, b, c, d, e, f, g, h is called *jumbled* if the letters a, b, c are not in order and the letters d, e are not in order. For example, $fbg\mathbf{d}ac\mathbf{e}h$ is *not* jumbled because d occurs before e, and $f\mathbf{a}e\mathbf{b}gd\mathbf{c}h$ is not jumbled because a occurs before b and b occurs before c. On the other hand, the permutation $fbgaedch$ is jumbled.

Use **Inclusion-Exclusion** to compute the number of jumbled permutations of a, b, c, d, e, f, g, h, and **explain** your reasoning. Your numerical answer may involve factorials, binomial coefficients, etc., and may be left unevaluated.

Solutions that do not use Inclusion-Exclusion will not receive credit.