Final Examination, Unit 4

Your name: ____________________

- This exam is **closed book** except for two 2-sided cribsheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	20		
2	15		
3	15		
Total	50		

Unit 4: Number Theory, Counting

Problem 1 (Modular Arithmetic) (20 points).
Let $p(x)$ be an integer polynomial, that is, $p(x) = \sum_{i=0}^{d} c_i x^i$ where the coefficients c_i are integers.

(a) Explain why $p(k) \equiv p(\text{rem}(k,\ n)) \pmod{n}$ for all integers k, n where $n > 1$.

Now let

$$q(x) ::= (x^2 - 4)(x^2 - 9),$$

and let $q(\mathbb{N}) ::= \{q(0), q(1), q(2), \ldots\}$.

(b) Verify that 3 divides every element of $q(\mathbb{N})$. *Hint:* part (a)

(c) Verify that 4 divides every element of $q(\mathbb{N})$.

(d) Prove that $\gcd(q(\mathbb{N})) = 12$.

Problem 2 (Asymptotic Notation) (15 points).

(a) Define a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = \Theta(n^2)$ and $\text{NOT}(f(n) \sim n^2)$. Explain why your function satisfies these conditions.

$$f(n) =$$

(b) Define a function $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = O(n^2)$, $g(n) \neq \Theta(n^2)$, $g(n) \neq o(n^2)$, and $n = O(g(n))$, and explain why your example works.

$$g(n) =$$

Problem 3 (Counting) (15 points).
Formulas (1)–(8) below are the answers to questions (a)–(i) below, in no particular order. Some answers may be used multiple times or not at all. Enter the number of the correct solution in the box below each question. No explanations are required.

$$(1).\quad \frac{n!}{(n-m)!} \qquad (2).\quad \binom{n+m}{m} \qquad (3).\quad (n-m)! \qquad (4).\quad m^n$$

$$(5).\quad \binom{n-1+m}{m} \qquad (6).\quad \binom{n-1+m}{n} \qquad (7).\quad 2^{mn} \qquad (8).\quad n^m$$

(a) How many solutions over the nonnegative integers are there to the equation $x_1 + x_2 + \cdots + x_n = m$?

(b) How many total functions[1] are there from a set of size m to a set of size n?

(c) How many length m words can be formed from an n-letter alphabet, if no letter is used more than once?

(d) How many length m words can be formed from an n-letter alphabet, if letters can be reused?

(e) How many binary relations are there from a set of size n to a set of size m?

(f) How many total injective[2] functions are there from set of size m to a set of size n where $n \geq m$?

(g) How many ways are there to place a total of m distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls?

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[1] Recall that *total function* means *exactly one arrow out*.
[2] Recall that *injective* means *at most one arrow in*.

(h) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn?

(i) How many ways are there to place a total of m **in**distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls?