Final Examination, Unit 2

Your name: ________________________________

- This exam is closed book except for two 2-sided crib sheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Grade</th>
<th>Grader</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Problem 1 (State Machines, Induction) (30 points).

Token replacing-1-3 is a single player game using a set of tokens, each colored black or white. In each move, a player can replace a black token with three white tokens, or replace a white token with three black tokens. We can model this game as a state machine whose states are pairs \((b, w)\) of nonnegative integers, where \(b\) is the number of black tokens and \(w\) the number of white ones.

The game has two possible start states: \((5, 4)\) or \((4, 3)\). We call a state reachable if it is reachable from at least one of the two start states.

We call a state \((b, w)\) eligible when

\[
\text{rem}(b - w, 4) = 1, \text{ AND } \min\{b, w\} \geq 3. \tag{2}
\]

(Recall that \text{rem}(n, 4) denotes the number \(0 \leq r \leq 3\) such that \(n = 4q + r\) for some \(q \in \mathbb{Z}\).)

This problem examines the connection between eligible states and reachable states.

(a) Give an example of a reachable state that is not eligible. Explain.

(b) Show that the derived variable \(b + w\) is strictly increasing. Conclude that state \((3, 2)\) is not reachable.

(e) Verify that \(\text{rem}(3b + w, 8)\) is a derived variable that is constant. Conclude that no state is reachable from both start states.
(d) Suppose \((b, w)\) is eligible and \(b \geq 6\). Verify that \((b - 3, w + 1)\) is eligible.

We now prove that every eligible state is reachable. For the rest of the problem, you may—and should—assume the following Fact:

**Fact.** If \(\max\{b, w\} \leq 5\) and \((b, w)\) is eligible, then \((b, w)\) is reachable.

(This is easy to verify since there are only nine states with \(b, w \in \{3, 4, 5\}\), but don’t waste time doing this.)

(e) Define the predicate \(P(n)\) to be:

\[ \forall (b, w).[b + w = n \text{ AND } (b, w) \text{ is eligible}] \text{ IMPLIES } (b, w) \text{ is reachable}. \]

Prove **carefully by strong induction** that \(P(n)\) is true for every integer \(n \geq 0\). **Hint:** Prove and use the fact that \(P(n - 1)\) IMPLIES \(P(n + 1)\).
Problem 2 (Relations, Predicates) (20 points). (a) Let $R$ be a binary relation on a set $D$. Each of the following formulas expresses the fact that $R$ has a familiar relational property such as reflexivity, asymmetry, transitivity, etc. For each of the five predicate formulas below, identify the name of that property.

(1) $\forall c, d. \ c \ R \ d \ \text{IFF} \ d \ R \ c$

(2) $\forall d. \ \text{NOT}(d \ R \ d)$

(3) $\forall c, d. \ c \ R \ d \ \text{IMPLIES} \ \text{NOT}(d \ R \ c)$

(4) $\forall b, d. \ [\exists c. (b \ R \ c \ \text{AND} \ c \ R \ d)] \ \text{IMPLIES} \ b \ R \ d$

(5) $\forall c \neq d. \ c \ R \ d \ \text{IMPLIES} \ \text{NOT}(d \ R \ c)$

(b) Below are five relational formulas encoding the same five familiar properties as in part (a), in scrambled order. Match these five relational formulas to the predicate formulas from part (a) by writing the number that each corresponds to. Your answers should simply be the numbers 1–5 in some order.

(6) $R \cap \text{Id}_D = \emptyset$

(7) $R \subseteq R^{-1}$

(8) $R \circ R \subseteq R$

(9) $R \cap R^{-1} \subseteq \text{Id}_D$

(10) $R \cap R^{-1} = \emptyset$

In these formulas, $\text{Id}_D$ is the “identity relation” on $D$, defined by $\text{Id}_D := \{(d, d) \mid d \in D\}$. 