Mathematics for Computer Science

MIT 6.042J/18.062J

Variance

Variance of an Indicator

I an indicator with $E[I]=p$:

$$\mathrm{Var}[I] ::= E[(I-p)^2] = E[I^2 - 2pI + p^2]$$
$$= E[I^2] - 2pE[I] + p^2$$
$$= E[I] - 2p \cdot p + p^2$$
$$= p - 2p^2 + p^2$$
$$= p - p^2 = pq$$

Calculating Variance

$$\mathrm{Var}[aR+b] = a^2\,\mathrm{Var}[R]$$
$$\mathrm{Var}[R] = E[R^2] - (E[R])^2$$

Calculating Variance

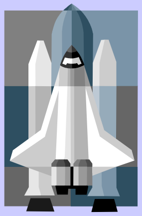
$$\mathrm{Var}[aR+b] = a^2\,\mathrm{Var}[R]$$
$$\mathrm{Var}[R] = E[R^2] - E^2[R]$$

simple proofs applying linearity of E[] to the def of Var[]

proof of 2nd Variance Formula

$$
\begin{aligned}
\boxed{Var[R]} &::= E[(R - \mu)^2] \\
&= E[R^2 - 2\mu R + \mu^2] \\
&= E[R^2] - 2\mu \cdot E[R] + E[\mu^2] \\
&= E[R^2] - 2\mu \cdot \mu + \mu^2 \\
&= E[R^2] - \mu^2 \\
&\boxed{= E[R^2] - E^2[R]}
\end{aligned}
$$

Space Station Mir



Destructs with probability p in any given hour

$E[F] = 1/p$ (Mean Time to Fail)

$Var[F] = ?$

Variance of Time to Failure

$$Pr[F = k] = q^{k-1}p$$

$$Var[F] = E[F^2] - E^2[F]$$

$$F = 1, 2, 3, \ldots, k, \ldots$$

$$F^2 = 1, 4, 9, \ldots, k^2, \ldots$$

Variance of Time to Failure

$$
\begin{aligned}
E[F^2] &::= \sum_{k=1}^{\infty} k^2 \cdot Pr[F^2 = k^2] \\
&= \sum_{k=1}^{\infty} k^2 \cdot Pr[F = k] \\
&= \frac{p}{q} \underbrace{\sum_{k=0}^{\infty} k^2 q^k}_{\text{has closed form}}
\end{aligned}
$$

Variance of Time to Failure

total expectation approach: $E[F^2] =$

$$E[F^2 \mid F = 1] \cdot Pr[F = 1]$$
$$+ E[F^2 \mid F > 1] \cdot Pr[F > 1]$$

Conditional time to failure

lemma: For F = time to failure, $g : \mathbb{R} \to \mathbb{R}$,

$$E[g(F) \mid F > n]$$

Conditional time to failure

lemma: For F = time to failure, $g : \mathbb{R} \to \mathbb{R}$,

$$E[g(F) \mid F > n] = E[g(F + n)]$$

Corollary:

$$E[F^2 \mid F > 1] = E[(F + 1)^2]$$

Variance of Time to Failure

total expectation approach: $E[F^2] =$

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$$+ E[F^2 \mid F > 1] \cdot Pr[F > 1]$$

Variance of Time to Failure

total expectation approach:

$$E[F^2] = 1 \cdot p + E[F^2 \mid F > 1] \cdot \Pr[F > 1]$$

Variance of Time to Failure

total expectation approach:

$$E[F^2] = 1 \cdot p + E[(F+1)^2] \cdot q$$

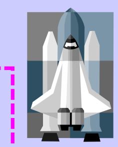
Variance of Time to Failure

total expectation approach:

$$E[F^2] = 1 \cdot p + (E[F^2] + 2/p + 1) \cdot q$$

now solve for $E[F^2]$

Mean Time to Failure



$$\mathrm{Var}[F] = \frac{1}{p}\left(\frac{1}{p} - 1\right)$$

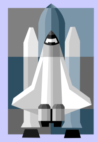
Mir1:

$p = 10^{-4}$, $E[F] = 10^4$, $\sigma < 10^4$

so by Chebyshev

$\Pr[\text{lasts} \geq 4 \cdot 10^4 \text{ hours}] =$

Mean Time to Failure



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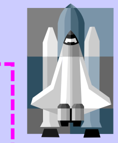
so by Chebyshev

$\Pr[F - 10^4 \geq 3 \cdot 10^4 \text{ hours}] \leq$

Mean Time to Failure



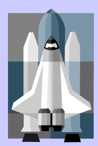
Mir1: $$\mathrm{Var}[F] = \frac{1}{p}\left(\frac{1}{p} - 1\right)$$

$p = 10^{-4}$, $E[F] = 10^4$, $\sigma < 10^4$

so by Chebyshev

$\Pr[|F - E[F]|] \geq 3 \cdot \sigma \text{ hours}] \leq 1/9$

Mean Time to Failure



Mir1: $$\mathrm{Var}[F] = \frac{1}{p}\left(\frac{1}{p} - 1\right)$$

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so by Chebyshev

$\Pr[\text{lasts} \geq 4.6 \text{ years}] \leq 1/9$

Calculating Variance

Pairwise Independent Additivity

$$\mathrm{Var}[R_1 + R_2 + \cdots + R_n]$$
$$= \mathrm{Var}[R_1] + \mathrm{Var}[R_2] + \cdots + \mathrm{Var}[R_n]$$

providing $R_1, R_2, \ldots, R_n$ are pairwise independent

again, a simple proof applying linearity of E[] to the def of Var[]