Variance of an Indicator

I an indicator with $E[I]=p$:


$$= E[I^2] - 2pE[I] + p^2$$

$$= E[I] - 2p \cdot p + p^2$$

$$= p - 2p^2 + p^2$$

$$= p - p^2 = pq$$

Calculating Variance

$$\text{Var} [aR + b] = a^2 \text{Var}[R]$$

$$\text{Var}[R] = E[R^2] - (E[R])^2$$

simple proofs applying linearity of $E[]$ to the def of $\text{Var}[]$
**Proof of 2nd Variance Formula**

\[
\text{Var}[R] := E[(R - \mu)^2] = E[R^2 - 2\mu R + \mu^2] = E[R^2] - 2\mu E[R] + E[\mu^2] = E[R^2] - 2\mu \cdot \mu + \mu^2 = E[R^2] - \mu^2 = E[R^2] - E^2[R]
\]

**Space Station Mir**

Destructs with probability \( p \) in any given hour

\[
E[F] = 1/p \quad \text{(Mean Time to Fail)}
\]

\[
\text{Var}[F] = ?
\]

**Variance of Time to Failure**

\[
\text{Pr}[F = k] = q^{k-1}p \quad F = 1, 2, 3, \ldots, k, \ldots
\]

\[
\text{Var}[F] = E[F^2] - E^2[F] = 1, 4, 9, \ldots, k^2, \ldots
\]

**Variance of Time to Failure**

\[
E[F^2] := \sum_{k=1}^{\infty} k^2 \cdot \text{Pr}[F^2 = k^2] = \sum_{k=1}^{\infty} k^2 \cdot \text{Pr}[F = k] = \frac{p}{q} \sum_{k=0}^{\infty} k^2 q^k
\]

has closed form
Variance of Time to Failure

Total expectation approach:

\[
E[F^2] = E[F^2 | F = 1] \cdot Pr[F = 1] + E[F^2 | F > 1] \cdot Pr[F > 1]
\]

Conditional time to failure

Lemma: For \( F = \) time to failure, \( g: \mathbb{R} \rightarrow \mathbb{R} \),

\[
E[g(F) | F > n] = E[g(F + n)]
\]

Corollary:

\[
E[F^2 | F > 1] = E[(F + 1)^2]
\]
Variance of Time to Failure

total expectation: \( E[F^2] = \)

- \( 1 \cdot p \)
- \( + E[F^2 | F > 1] \cdot \Pr[F > 1] \)

Variance of Time to Failure

total expectation: \( E[F^2] = \)

- \( 1 \cdot p \)
- \( + E[(F + 1)^2] \cdot q \)

Variance of Time to Failure

total expectation: \( E[F^2] = \)

- \( 1 \cdot p \)
- \( + (E[F^2] + 2/p + 1) \cdot q \)

now solve for \( E[F^2] \)

Mean Time to Failure

\[ \text{Mir1: } \]
\[ \text{Var}[F] = \frac{1}{p} \left( \frac{1}{p} - 1 \right) \]

\[ p = 10^{-4}, \ E[F] = 10^4, \ \sigma < 10^4 \]

so by Chebyshev

\[ \Pr[\text{lasts } \geq 4 \cdot 10^4 \text{ hours}] = \]
Mean Time to Failure

\[ \text{Var}[F] = \frac{1}{p} \left( \frac{1}{p} - 1 \right) \]

Mir1:

\[ p = 10^{-4}, E[F] = 10^4, \sigma < 10^4 \]

so by Chebyshev

\[ \text{Pr}[F - 10^4 \geq 3 \cdot 10^4 \text{ hours}] \leq \]

Calculating Variance

Pairwise Independent Additivity

\[ \text{Var}[R_1 + R_2 + \cdots + R_n] \]

\[ = \text{Var}[R_1] + \text{Var}[R_2] + \cdots + \text{Var}[R_n] \]

providing \( R_1, R_2, \ldots, R_n \) are pairwise independent

again, a simple proof applying linearity of \( \text{E}[\cdot] \) to the def of \( \text{Var}[\cdot] \)