In-Class Problems Week 6, Wed.

Problem 1.
Let \( \{1, 2, 3\}^\omega \) be the set of infinite sequences containing only the numbers 1, 2, and 3. For example, some sequences of this kind are:

\[
\begin{align*}
\langle 1, 1, 1 \ldots \rangle, \\
\langle 2, 2, 2 \ldots \rangle, \\
\langle 3, 2, 1 \ldots \rangle.
\end{align*}
\]

Give two proofs that \( \{1, 2, 3\}^\omega \) is uncountable:
(a) by showing \( \{1, 2, 3\}^\omega \) surj \( S \) for some known uncountable set \( S \), and
(b) by a direct diagonalization argument.

Problem 2.
You don’t really have to go down the diagonal in a “diagonal” argument.
Let’s review the historic application of a diagonal argument to one-way infinite sequences

\[
\langle e_0, e_1, e_2, \ldots, e_k, \ldots \rangle.
\]

The angle brackets appear above as a reminder that the sequence is not a set: its elements appear in order, and the same element may appear multiple times.\(^1\)

The general setup for a diagonal argument is that we have some sequence \( S \) whose elements are themselves one-way infinite sequences. We picture the sequence \( S = \langle s_0, s_1, s_2, \ldots \rangle \) running vertically downward, and each sequence \( s_k \in S \) running horizontally to the right:

\[
s_k = \langle s_{k,0}, s_{k,1}, s_{k,2}, \ldots \rangle.
\]

So we have a 2-D matrix that is infinite down and to the right:

\[
\begin{array}{cccccc}
0 & 1 & 2 & \ldots & k & \ldots \\
 s_0 & s_{0,0} & s_{0,1} & s_{0,2} & \ldots & s_{0,k} & \ldots \\
 s_1 & s_{1,0} & s_{1,1} & s_{1,2} & \ldots & s_{1,k} & \ldots \\
 s_2 & s_{2,0} & s_{2,1} & s_{2,2} & \ldots & s_{2,k} & \ldots \\
 \vdots & & & & & & \\
 s_k & & & & & & \ldots \end{array}
\]

\(^1\)The right angle-bracket is not really visible, since the sequence does not have a right end. If such one-way infinite sequences seem worrisome, you can replace them with total functions on \( \mathbb{N} \). So the sequence above simply becomes the function \( e(n) := e_n \).
The diagonal argument explains how to find a “new” sequence, that is, a sequence that is not in \( S \). Namely, create the new sequence by going down the diagonal of the matrix and, for each element encountered, add a differing element to the sequence being created. In other words, the diagonal sequence is

\[
D_S := \langle \bar{s}_0, 0, \bar{s}_1, 1, \bar{s}_2, 2, \ldots, \bar{s}_k, k, \ldots \rangle
\]

where \( \bar{x} \) indicates some element that is not equal to \( x \). Now \( D_S \) is a sequence that is not in \( S \) because it differs from every sequence in \( S \), namely, it differs from the \( k \)th sequence in \( S \) at position \( k \).

For definiteness, let’s say\(^1\)

\[
\bar{x} := \begin{cases} 
1 & \text{if } x \neq 1, \\
2 & \text{otherwise.}
\end{cases}
\]

With this contrivance, we have gotten the diagonal sequence \( D_S \) to be a sequence of 1’s and 2’s that is not in \( S \).

But as we said at the beginning, you don’t have to go down the diagonal. You could, for example, follow a line with a slope of \(-1/4\) to get a new sequence

\[
T_S := \langle \bar{s}_0, 0, t_1, 1, t_2, 2, \bar{s}_3, 3, t_4, 4, \bar{s}_5, 5, t_6, 6, \bar{s}_7, 7, \ldots, t_{4k-1}, 4k-1, \bar{s}_{4k}, 4k, t_{4k+1}, 4k+1, \ldots \rangle
\]

where the \( t_i \)’s can be anything at all. Any such \( T_S \) will be a new sequence because it will differ from every row of the matrix, but this time it differs from the \( k \)th row at position \( 4k \).

(a) By letting all the \( t_i \)’s be 2’s, we get a new sequence \( T_S \) of 1’s and 2’s whose elements (in the limit) are at least three-quarters equal to 2. Explain how to find an uncountable number of such sequences of 1’s and 2’s.

*Hint:* Use slope \(-1/8\) and fill six out of eight places with 2’s. The abbreviation

\[
2^{(n)} := 2, 2, \ldots, 2
\]

for a length-\( n \) sequence of 2’s may be helpful, in particular \( 2^{(6)} \).

(b) Let’s say a sequence has a negligible fraction of non-2 elements if, in the limit, it has a fraction of at most \( \varepsilon \) non-2 elements for *every* \( \varepsilon > 0 \).\(^2\) Describe how to define a sequence not in \( S \) that has a negligible fraction of non-2 elements.

(c) Optional: Describe how to find a sequence that differs infinitely many times from every sequence in \( S \).

*Hint:* Divide \( \mathbb{N} \) into an infinite number of non-overlapping infinite pieces.

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**Problem 3.**

For any sets \( A \) and \( B \), let \([A \to B]\) be the set of total functions from \( A \) to \( B \). Prove that if \( A \) is not empty and \( B \) has more than one element, then \( A \) strict \([A \to B]\).

*Hint:* Suppose that \( \sigma \) is a function from \( A \) to \([A \to B]\) mapping each element \( a \in A \) to a function \( \sigma_a : A \to B \). Suppose \( b, c \in B \) and \( b \neq c \). Then define

\[
\text{diag}(a) := \begin{cases} 
c & \text{if } \sigma_a(a) = b, \\
b & \text{otherwise.}
\end{cases}
\]

\(^2\)In other words, this means

\[
\lim_{n \to \infty} \frac{\text{(number of non-2s among the first } n \text{ elements)}}{n} = 0.
\]
Problem 4.

Let
\[ f_0, f_1, f_2, \ldots, f_k, \ldots \]
be an infinite sequence of positive real-valued total functions \( f_k : \mathbb{N} \to \mathbb{R}^+ \).

(a) A function from \( \mathbb{N} \) to \( \mathbb{R} \) that eventually gets bigger than every one of the \( f_k \) is said to majorize the set of \( f_k \)'s. So finding a majorizing function is a different way than diagonalization to find a function that is not equal to any of the \( f_k \)'s.

Given an explicit formula for a majorizing function \( g : \mathbb{N} \to \mathbb{R} \) for the \( f_k \)'s, and indicate how big \( n \) should be to ensure that \( g(n) > f_k(n) \).

(b) Modify your answer to part (a), if necessary, to define a majorizing function \( h \) that grows more rapidly than \( f_k \) for every \( k \in \mathbb{N} \), namely,
\[
\lim_{n \to \infty} \frac{f_k(n)}{h(n)} = 0.
\]
Explain why.

Supplemental (Optional)

Problem 5.

There are lots of different sizes of infinite sets. For example, starting with the infinite set \( \mathbb{N} \) of nonnegative integers, we can build the infinite sequence of sets
\[
\mathbb{N} \text{ strict } \text{pow}(\mathbb{N}) \text{ strict } \text{pow}(\text{pow}(\mathbb{N})) \text{ strict } \text{pow}(\text{pow}(\text{pow}(\mathbb{N}))) \text{ strict } \ldots
\]
where each set is “strictly smaller” than the next one by Theorem 8.1.12. Let \( \text{pow}^n(\mathbb{N}) \) be the \( n \)th set in the sequence, and
\[
U := \bigcup_{n=0}^{\infty} \text{pow}^n(\mathbb{N}).
\]

(a) Prove that
\[ U \text{ surj } \text{pow}^n(\mathbb{N}). \] (1)
for all \( n > 0 \).

(b) Prove that
\[ \text{pow}^n(\mathbb{N}) \text{ strict } U \]
for all \( n \in \mathbb{N} \).

Now of course, we could take \( U, \text{pow}(U), \text{pow}(\text{pow}(U)), \ldots \) and keep on in this way building still bigger infinities indefinitely.