In-Class Problems Week 6, Wed.

Problem 1.
Let $\{1, 2, 3\}^{\omega}$ be the set of infinite sequences containing only the numbers 1, 2, and 3. For example, some sequences of this kind are:

$$\langle 1, 1, 1, 1 \ldots \rangle,$$
$$\langle 2, 2, 2, 2 \ldots \rangle,$$
$$\langle 3, 2, 1, 3 \ldots \rangle.$$

Give two proofs that $\{1, 2, 3\}^{\omega}$ is uncountable:

(a) by showing $\{1, 2, 3\}^{\omega}$ surj S for some known uncountable set S, and

(b) by a direct diagonalization argument.

Problem 2.
You don't really have to go down the diagonal in a "diagonal" argument.

Let's review the historic application of a diagonal argument to one-way infinite sequences

$$\langle e_0, e_1, e_2, \ldots, e_k, \ldots \rangle.$$

The angle brackets appear above as a reminder that the sequence is not a set: its elements appear in order, and the same element may appear multiple times.[1]

The general setup for a diagonal argument is that we have some sequence S whose elements are themselves one-way infinite sequences. We picture the sequence $S = \langle s_0, s_1, s_2, \ldots \rangle$ running vertically downward, and each sequence $s_k \in S$ running horizontally to the right:

$$s_k = \langle s_{k,0}, s_{k,1}, s_{k,2}, \ldots \rangle.$$

So we have a 2-D matrix that is infinite down and to the right:

	0	1	2	$\ldots$	$k \ldots$
s_0	$s_{0,0}$	$s_{0,1}$	$s_{0,2}$	$\ldots$	$s_{0,k} \cdots$
s_1	$s_{1,0}$	$s_{1,1}$	$s_{1,2}$	$\ldots$	$s_{1,k} \cdots$
s_2	$s_{2,0}$	$s_{2,1}$	$s_{2,2}$	$\ldots$	$s_{2,k} \cdots$
$\vdots$				$\vdots$	
s_k				$\ldots$	$s_{k.k} \cdots$

[1]The right angle-bracket is not really visible, since the sequence does not have a right end. If such one-way infinite sequences seem worrisome, you can replace them with total functions on $\mathbb{N}$. So the sequence above simply becomes the function e on $\mathbb{N}$ where $e(n) ::= e_n$.

The diagonal argument explains how to find a "new" sequence, that is, a sequence that is not in S. Namely, create the new sequence by going down the diagonal of the matrix and, for each element encountered, add a differing element to the sequence being created. In other words, the diagonal sequence is

$$D_S ::= \langle \overline{s_{0,0}},\ \overline{s_{1,1}},\ \overline{s_{2,2}}, \dots, \overline{s_{k,k}}, \dots \rangle$$

where $\overline{x}$ indicates some element that is not equal to x. Now D_S is a sequence that is not in S because it differs from every sequence in S, namely, it differs from the kth sequence in S at position k.

For definiteness, let's say

$$\overline{x} ::= \begin{cases} 1 & \text{if } x \neq 1, \\ 2 & \text{otherwise.} \end{cases}$$

With this contrivance, we have gotten the diagonal sequence D_S to be a sequence of 1's and 2's that is not in S.

But as we said at the beginning, you don't have to go down the diagonal. You could, for example, follow a line with a slope of $-1/4$ to get a new sequence

$$T_S ::= \langle \overline{s_{0,0}},\ t_1,\ t_2,\ t_3,\ \overline{s_{1,4}},\ t_5,\ t_6,\ t_7,\ \overline{s_{2,8}},\ \dots,\ t_{4k-1},\ \overline{s_{k,4k}},\ t_{4k+1},\ \dots \rangle$$

where the t_i's can be anything at all. Any such T_s will be a new sequence because it will differ from every row of the matrix, but this time it differs from the kth row at position $4k$.

(a) By letting all the t_i's be 2's, we get a new sequence T_S of 1's and 2's whose elements (in the limit) are at least three-quarters equal to 2. Explain how to find an *uncountable* number of such sequences of 1's and 2's.

Hint: Use slope $-1/8$ and fill six out of eight places with 2's. The abbreviation

$$2^{(n)} ::= \underbrace{2, 2, \dots, 2}_{\text{length } n}$$

for a length-n sequence of 2's may be helpful, in particular $2^{(6)}$.

(b) Let's say a sequence has a *negligible fraction of non-2 elements* if, in the limit, it has a fraction of at most ϵ non-2 elements for *every* $\epsilon > 0$.[2] Describe how to define a sequence not in S that has a *negligible fraction of non-2 elements*.

(c) Optional: Describe how to find a sequence that differs *infinitely many times* from every sequence in S.

Hint: Divide $\mathbb{N}$ into an infinite number of non-overlapping infinite pieces.

Problem 3.

For any sets A and B, let $[A \to B]$ be the set of total functions from A to B. Prove that if A is not empty and B has more than one element, then A strict $[A \to B]$.

Hint: Suppose that σ is a function from A to $[A \to B]$ mapping each element $a \in A$ to a function $\sigma_a : A \to B$. Suppose $b, c \in B$ and $b \neq c$. Then define

$$\text{diag}(a) ::= \begin{cases} c \text{ if } \sigma_a(a) = b, \\ b \text{ otherwise.} \end{cases}$$

[2]In other words, this means

$$\lim_{n\to\infty} (\text{number of non-2s among the first } n \text{ elements})/n = 0.$$

Problem 4.
Let

$$f_0, f_1, f_2, \ldots, f_k, \ldots$$

be an infinite sequence of positive real-valued total functions $f_k : \mathbb{N} \to \mathbb{R}^+$.

(a) A function from $\mathbb{N}$ to $\mathbb{R}$ that eventually gets bigger that every one of the f_k is said to *majorize* the set of f_k's. So finding a majorizing function is a different way than diagonalization to find a function that is not equal to any of the f_k's.

Given an explicit formula for a majorizing function $g : \mathbb{N} \to \mathbb{R}$ for the f_k's, and indicate how big n should be to ensure that $g(n) > f_k(n)$.

(b) Modify your answer to part (a), if necessary, to define a majorizing function h that *grows* more rapidly than f_k for every $k \in \mathbb{N}$, namely,

$$\lim_{n \to \infty} \frac{f_k(n)}{h(n)} = 0.$$

Explain why.

Supplemental (Optional)

Problem 5.
There are lots of different sizes of infinite sets. For example, starting with the infinite set $\mathbb{N}$ of nonnegative integers, we can build the infinite sequence of sets

$$\mathbb{N} \text{ strict } \text{pow}(\mathbb{N}) \text{ strict } \text{pow}(\text{pow}(\mathbb{N})) \text{ strict } \text{pow}(\text{pow}(\text{pow}(\mathbb{N}))) \text{ strict } \ldots.$$

where each set is "strictly smaller" than the next one by Theorem 8.1.12. Let $\text{pow}^n(\mathbb{N})$ be the nth set in the sequence, and

$$U ::= \bigcup_{n=0}^{\infty} \text{pow}^n(\mathbb{N}).$$

(a) Prove that

$$U \text{ surj } \text{pow}^n(\mathbb{N}), \tag{1}$$

for all $n > 0$.

(b) Prove that

$$\text{pow}^n(\mathbb{N}) \text{ strict } U$$

for all $n \in \mathbb{N}$.

Now of course, we could take $U, \text{pow}(U), \text{pow}(\text{pow}(U)), \ldots$ and keep on in this way building still bigger infinities indefinitely.