In-Class Problems Week 6, Mon.

Problem 1.
If set S is infinite and there is a surjective function ([≤ 1 out, ≥ 1 in] mapping) $f : \mathbb{N} \to S$, prove that S is countable by explicitly constructing a bijection $g : \mathbb{N} \to S$. (This proves the more challenging part of Lemma 8.1.9, so please only rely on Definition 8.1.8.)

Hint: The "sequence" $f(0), f(1), f(2), \ldots$ may have undefined and/or repeated values. How can you filter these out?

Problem 2.
The rational numbers fill the space between integers, so a first thought is that there must be more of them than the integers, but it's not true. In this problem you'll show that there are the same number of positive rationals as positive integers. That is, the positive rationals are countable.

(a) Define a bijection between the set $\mathbb{Z}^+$ of positive integers, and the set $(\mathbb{Z}^+ \times \mathbb{Z}^+)$ of all pairs of positive integers:

$$\begin{array}{l}
(1,1), (1,2), (1,3), (1,4), (1,5), \ldots \\
(2,1), (2,2), (2,3), (2,4), (2,5), \ldots \\
(3,1), (3,2), (3,3), (3,4), (3,5), \ldots \\
(4,1), (4,2), (4,3), (4,4), (4,5), \ldots \\
(5,1), (5,2), (5,3), (5,4), (5,5), \ldots \\
\qquad \vdots
\end{array}$$

(b) Conclude that the set $\mathbb{Q}^+$ of all positive rational numbers is countable.

Hint: Use Problem 1.

Problem 3. **(a)** Several students felt the proof of Lemma 8.1.7 was worrisome, if not circular. What do you think?

Lemma 8.1.7. Let A be a set and $b \notin A$. If A is infinite, then there is a bijection from $A \cup \{b\}$ to A.

Proof. Here's how to define the bijection: since A is infinite, it certainly has at least one element; call it a_0. But since A is infinite, it has at least two elements, and one of them must not be equal to a_0; call this new element a_1. But since A is infinite, it has at least three elements, one of which must not equal a_0 or a_1; call this new element a_2. Continuing in the way, we conclude that there is an infinite sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of different elements of A. Now we can define a bijection $f : A \cup \{b\} \to A$:

$$\begin{aligned} f(b) &::= a_0, \\ f(a_n) &::= a_{n+1} && \text{for } n \in \mathbb{N}, \\ f(a) &::= a && \text{for } a \in A - \{a_0, a_1, \ldots\}. \end{aligned}$$

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(b) Use the proof of Lemma 8.1.7 to show that if A is an infinite set, then A surj $\mathbb{N}$, that is, every infinite set is "at least as big as" the set of nonnegative integers.

Problem 4.

In this problem you will prove a fact that may surprise you—or make you even more convinced that set theory is nonsense: the half-open unit interval is actually the "*same size*" as the nonnegative quadrant of the real plane![1] Namely, there is a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$.

(a) Describe a bijection from $(0, 1]$ to $[0, \infty)$.

Hint: $1/x$ almost works.

(b) An infinite sequence of the decimal digits $\{0, 1, \ldots, 9\}$ will be called *long* if it does not end with all 0's. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all such long sequences. Describe a bijection from L to the half-open real interval $(0, 1]$.

Hint: Put a decimal point at the beginning of the sequence.

(c) Describe a surjective function from L to L^2 that involves alternating digits from two long sequences. *Hint:* The surjection need not be total.

(d) Prove the following lemma and use it to conclude that there is a bijection from L^2 to $(0, 1]^2$.

Lemma. *Let A and B be nonempty sets. If there is a bijection from A to B, then there is also a bijection from $A \times A$ to $B \times B$.*

(e) Conclude from the previous parts that there is a surjection from $(0, 1]$ to $(0, 1]^2$. Then appeal to the Schröder-Bernstein Theorem to show that there is actually a bijection from $(0, 1]$ to $(0, 1]^2$.

(f) Complete the proof that there is a bijection from $(0, 1]$ to $[0, \infty)^2$.

Supplemental (Optional)

Problem 5.

This problem provides a proof of the [Schröder-Bernstein] Theorem:

$$\text{If } A \text{ inj } B \text{ and } B \text{ inj } A, \text{ then } A \text{ bij } B. \tag{1}$$

[1] The half-open unit interval $(0, 1]$ is $\{r \in \mathbb{R} \mid 0 < r \le 1\}$. Similarly, $[0, \infty) ::= \{r \in \mathbb{R} \mid r \ge 0\}$.

Since A inj B and B inj A, there are are total injective functions $f : A \to B$ and $g : B \to A$.

Assume for simplicity that A and B have no elements in common. Let's picture the elements of A arranged in a column, and likewise B arranged in a second column to the right, with left-to-right arrows connecting a to $f(a)$ for each $a \in A$ and likewise right-to-left arrows for g. Since f and g are total functions, there is *exactly one* arrow *out* of each element. Also, since f and g are injections, there is *at most one* arrow *into* any element.

So starting at any element, there is a unique and unending path of arrows going forwards (it might repeat). There is also a unique path of arrows going backwards, which might be unending, or might end at an element that has no arrow into it. These paths are completely separate: if two ran into each other, there would be two arrows into the element where they ran together.

This divides all the elements into separate paths of four kinds:

(i) paths that are infinite in both directions,

(ii) paths that are infinite going forwards starting from some element of A.

(iii) paths that are infinite going forwards starting from some element of B.

(iv) paths that are unending but finite.

(a) What do the paths of the last type (iv) look like?

(b) Show that for each type of path, either

(i) the f-arrows define a bijection between the A and B elements on the path, or

(ii) the g-arrows define a bijection between B and A elements on the path, or

(iii) both sets of arrows define bijections.

For which kinds of paths do both sets of arrows define bijections?

(c) Explain how to piece these bijections together to form a bijection between A and B.

(d) Justify the assumption that A and B are disjoint.