In-Class Problems Week 4, Wed.

Problem 1.
Any amount of 12 or more cents postage can be made using only 3¢ and 7¢ stamps. Prove this *by induction* using the induction hypothesis

$$S(n) ::= n + 12 \text{ cents postage can be made using only 3¢ and 7¢ stamps.}$$

Problem 2.
The following Lemma is true, but the *proof* given for it below is defective. Pinpoint *exactly* where the proof first makes an unjustified step and explain why it is unjustified.

Lemma. *For any prime p and positive integers $n, x_1, x_2, \dots, x_n$, if $p \mid x_1 x_2 \dots x_n$, then $p \mid x_i$ for some $1 \le i \le n$.*

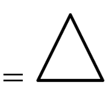
Bogus proof. Proof by strong induction on n. The induction hypothesis $P(n)$ is that Lemma holds for n.

Base case $n = 1$: When $n = 1$, we have $p \mid x_1$, therefore we can let $i = 1$ and conclude $p \mid x_i$.

Induction step: Now assuming the claim holds for all $k \le n$, we must prove it for $n + 1$.

So suppose $p \mid x_1 x_2 \cdots x_{n+1}$. Let $y_n = x_n x_{n+1}$, so $x_1 x_2 \cdots x_{n+1} = x_1 x_2 \cdots x_{n-1} y_n$. Since the right-hand side of this equality is a product of n terms, we have by induction that p divides one of them. If $p \mid x_i$ for some $i < n$, then we have the desired i. Otherwise $p \mid y_n$. But since y_n is a product of the two terms x_n, x_{n+1}, we have by strong induction that p divides one of them. So in this case $p \mid x_i$ for $i = n$ or $i = n + 1$. ■

Problem 3.
Divided Equilateral Triangles[1] (DETs) can be built up as follows:

- A single equilateral triangle counts as a DET whose only unit subtriangle is itself.
- If $T ::=$  is a DET, then the equilateral triangle T' built out of four copies of T as shown in in Figure 1 is also a DET, and the unit subtriangles of T' are exactly the unit subtriangles of each of the copies of T.

From here on "subtriangle" will mean unit subtriangle.

(a) Define the *length* of a DET to be the number of subtriangles with an edge on its base. Prove **by induction on length** that the total number of subtriangles of a DET is the square of its length.

(b) Show that a DET with one of its corner subtriangles removed can be tiled with trapezoids built out of three subtriangles as in Figure 2.

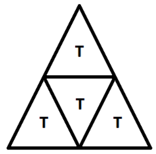


Figure 1 DET T' from Four Copies of DET T

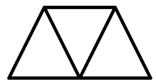


Figure 2 Trapezoid from Three Triangles

Problem 4.
The Block Stacking Game[2] goes as follows: You begin with a stack of n boxes. Then you make a sequence of moves. In each move, you divide one stack of boxes into two nonempty stacks. The game ends when you have n stacks, each containing a single box. You earn points for each move; in particular, if you divide one stack of height $a + b$ into two stacks with heights a and b, then you score ab points for that move. Your overall score is the sum of the points that you earn for each move. What strategy should you use to maximize your total score?

Define the *potential* $p(S)$ of a stack of blocks S to be $k(k-1)/2$ where k is the number of blocks in S. Define the potential $p(A)$ of a set of stacks A to be the sum of the potentials of the stacks in A.

Show that for any set of stacks A if a sequence of moves starting with A leads to another set of stacks B then $p(A) \geq p(B)$, and the score for this sequence of moves is $p(A) - p(B)$.

Hint: Try induction on the number of moves to get from A to B.

Problem 5.
Find the flaw in the following bogus proof that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Bogus proof. The proof is by induction on n, with hypothesis

$$P(n) ::= \forall k \leq n. \, a^k = 1,$$

where k is a nonnegative integer valued variable.

Base Case: $(n = 0)$. $P(0)$ is equivalent to $a^0 = 1$, which is true by definition of a^0. (By convention, this holds even if $a = 0$.)

Inductive Step: By induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq n$. But then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$

which implies that $P(n+1)$ holds. It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$, and in particular, $a^n = 1$ holds for all $n \in \mathbb{N}$.

■

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[1] Adapted from [49].

[2] Excerpted from Section 5.2.5.

Stack Heights											**Score**
$\underline{10}$											
5	$\underline{5}$										25 points
$\underline{5}$	3	2									6
$\underline{4}$	3	2	1								4
2	$\underline{3}$	2	1	2							4
$\underline{2}$	2	2	1	2	1						2
1	$\underline{2}$	2	1	2	1	1					1
1	1	$\underline{2}$	1	2	1	1	1				1
1	1	1	1	$\underline{2}$	1	1	1	1			1
1	1	1	1	1	1	1	1	1	1		1
							Total Score			=	45 points

Figure 3 An example of the stacking game with $n = 10$ boxes. On each line, the underlined stack is divided in the next step.