In-Class Problems Week 4, Mon.

Problem 1.
The inverse $R^{-1}$ of a binary relation $R$ from $A$ to $B$ is the relation from $B$ to $A$ defined by:

$$b \, R^{-1} \, a \iff a \, R \, b.$$ 

In other words, you get the diagram for $R^{-1}$ from $R$ by “reversing the arrows” in the diagram describing $R$. Now many of the relational properties of $R$ correspond to different properties of $R^{-1}$. For example, $R$ is total iff $R^{-1}$ is a surjection.

Fill in the remaining entries in this table:

<table>
<thead>
<tr>
<th>$R$ is</th>
<th>$R^{-1}$ is</th>
</tr>
</thead>
<tbody>
<tr>
<td>total</td>
<td>a surjection</td>
</tr>
<tr>
<td>a function</td>
<td></td>
</tr>
<tr>
<td>a surjection</td>
<td></td>
</tr>
<tr>
<td>an injection</td>
<td></td>
</tr>
<tr>
<td>a bijection</td>
<td></td>
</tr>
</tbody>
</table>

*Hint:* Explain what’s going on in terms of “arrows” from $A$ to $B$ in the diagram for $R$.

### Arrow Properties

**Definition.** A binary relation, $R$ is

- a *function* when it has the $[\leq 1 \text{ arrow out}]$ property.
- a *surjective* when it has the $[\geq 1 \text{ arrows in}]$ property. That is, every point in the right-hand, codomain column has at least one arrow pointing to it.
- a *total* when it has the $[\geq 1 \text{ arrows out}]$ property.
- a *injective* when it has the $[\leq 1 \text{ arrow in}]$ property.
- a *bijective* when it has both the $[\leq 1 \text{ arrow out}]$ and the $[\geq 1 \text{ arrow in}]$ property.

Problem 2.
Assume $f : A \to B$ is total function, and $A$ is finite. Replace the $\ast$ with one of $\leq, =, \geq$ to produce the strongest correct version of the following statements:

(a) $|f(A)| \ast |B|$.

(b) If $f$ is a surjection, then $|A| \ast |B|$.

(c) If $f$ is a surjection, then $|f(A)| \ast |B|$.
(d) If $f$ is an injection, then $|f(A)| \star |A|$.

(e) If $f$ is a bijection, then $|A| \star |B|$.

**Problem 3.**

Let $R : A \rightarrow B$ be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule 1.

**Lemma.** If $R$ is a function, and $X \subseteq A$, then

$$|X| \geq |R(X)|.$$

**Problem 4.**

Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size $n$, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size $m$. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

**Problem 5.** (a) Prove that if $A \text{ surj } B$ and $B \text{ surj } C$, then $A \text{ surj } C$.

(b) Explain why $A \text{ surj } B$ iff $B \text{ inj } A$.

(c) Conclude from (a) and (b) that if $A \text{ inj } B$ and $B \text{ inj } C$, then $A \text{ inj } C$.

(d) According to the official definition, $A \text{ inj } B$ requires a total injective relation ($[\geq 1 \text{ out}, \leq 1 \text{ in}]$). Explain why $A \text{ inj } B$ iff there is a total injective function ($[[= 1 \text{ out}, \leq 1 \text{ in}])$ from $A$ to $B$.