In-Class Problems Week 2, Fri.

Problem 1.
For each of the logical formulas, indicate whether or not it is true when the domain of discourse is $\mathbb{N}$, (the nonnegative integers 0, 1, 2, ...), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers). Add a brief explanation to the few cases that merit one.

$$\exists x.\, x^2 = 2$$
$$\forall x.\exists y.\, x^2 = y$$
$$\forall y.\exists x.\, x^2 = y$$
$$\forall x \neq 0.\exists y.\, xy = 1$$
$$\exists x.\exists y.\, x + 2y = 2 \text{ AND } 2x + 4y = 5$$

Problem 2.
The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings: λ, 0, 1, 00, 01, 10, 11, 000, 001, ... (Here λ denotes the empty string.) In your translations, you may use all the ordinary logic symbols (including =), variables, and the binary symbols $\texttt{0}$, $\texttt{1}$ denoting 0, 1.

A string like $\texttt{01}x\texttt{0}y$ of binary symbols and variables denotes the *concatenation* of the symbols and the binary strings represented by the variables. For example, if the value of x is $\texttt{011}$ and the value of y is $\texttt{1111}$, then the value of $\texttt{01}x\texttt{0}y$ is the binary string $\texttt{0101101111}$.

Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate NO-1S below).

Meaning	Formula	Name
x is a prefix of y	$\exists z\ (xz = y)$	PREFIX(x, y)
x is a substring of y	$\exists u \exists v\ (uxv = y)$	SUBSTRING(x, y)
x is empty or a string of 0's	NOT(SUBSTRING$(\texttt{1}, x)$)	NO-1S(x)

(a) x consists of three copies of some string.

(b) x is an even-length string of 0's.

(c) x does not contain both a 0 and a 1.

(d) x is the binary representation of $2^k + 1$ for some integer $k \geq 0$.

(e) An elegant, slightly trickier way to define NO-1S(x) is:

$$\text{PREFIX}(x, \texttt{0}x). \tag{*}$$

Explain why (*) is true only when x is a string of 0's.

Problem 3.

The following predicate logic formula is invalid:

$$[\forall x, \exists y. P(x, y) \text{ IMPLIES } \exists y, \forall x. P(x, y) \tag{1}$$

Circle the items describing counter models for (1), and briefly explain.

1. The predicate $P(x, y) =$ '$y \cdot x = 1$' where the domain of discourse is $\mathbb{Q}$.
2. The predicate $P(x, y) =$ '$y < x$' where the domain of discourse is $\mathbb{R}$.
3. The predicate $P(x, y) =$ '$y \cdot x = 2$' where the domain of discourse is $\mathbb{R}$ without 0.
4. The predicate $P(x, y) =$ '$yxy = x$' where the domain of discourse is the set of all binary strings, including the empty string.

Problem 4.
Translate the following sentence into a predicate formula:

> There is a student who has e-mailed at most two other people in the class, besides possibly himself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are

- equality, and
- $E(x, y)$, meaning that "x has sent e-mail to y."

Supplemental Problem[1]

Problem 5. (a) A predicate R on the nonnegative integers is true *infinitely often (i.o.)* when $R(n)$ is true for infinitely many $n \in \mathbb{N}$.

We can express the fact that R is true i.o. with a formula of the form:

$$\mathbf{Q}_1 \ \ \mathbf{Q}_2. \ R(n),$$

where $\mathbf{Q}_1, \mathbf{Q}_2$ are quantifiers from among

$$\begin{array}{cccc} \forall n, & \exists n, & \forall n \geq n_0, & \exists n \geq n_0, \\ \forall n_0, & \exists n_0, & \forall n_0 \geq n, & \exists n_0 \geq n, \end{array}$$

and n, n_0 range over nonnegative integers.

Identify the proper quantifers: $\mathbf{Q}_1$____, $\mathbf{Q}_2$____

(b) A predicate S on the nonnegative integers is true *almost everywhere (a.e.)* when $S(n)$ is false for only finitely many $n \in \mathbb{N}$.

We can express the fact that S is true a.e. with a formula of the form

$$\mathbf{Q}_3 \ \ \mathbf{Q}_4. \ S(n),$$

where $\mathbf{Q}_3, \mathbf{Q}_4$ are quantifiers from those above.

Identify the proper quantifers: $\mathbf{Q}_3$____, $\mathbf{Q}_4$____

[1] There is no need to study supplemental problems when preparing for exams. They offer optional review or optional additional topics.