

## In-Class Problems Week 1, Fri.

### Problem 1.

Prove that if  $a \cdot b = n$ , then either  $a$  or  $b$  must be  $\leq \sqrt{n}$ , where  $a, b$ , and  $n$  are nonnegative real numbers.  
*Hint:* by contradiction, Section 1.8.

### Problem 2.

How far can you generalize the proof, repeated below, that  $\sqrt{2}$  is irrational?<sup>1</sup> For example, how about  $\sqrt{3}$ ?

**Theorem.**  $\sqrt{2}$  is an irrational number.

*Proof.* The proof is by contradiction: assume that  $\sqrt{2}$  is rational, that is,

$$\sqrt{2} = \frac{n}{d}, \tag{1}$$

where  $n$  and  $d$  are integers. Now consider the smallest such positive integer denominator  $d$ . We will prove in a moment that the numerator,  $n$ , and the denominator  $d$  are both even. This implies that

$$\frac{n/2}{d/2}$$

is a fraction equal to  $\sqrt{2}$  with a smaller positive integer denominator, a contradiction.

*Since the assumption that  $\sqrt{2}$  is rational leads to this contradiction, the assumption must be false. That is,  $\sqrt{2}$  is indeed irrational.* This italicized comment on the implication of the contradiction normally goes without saying, but since this is an early example of proof by contradiction, we've said it.

To prove that  $n$  and  $d$  have 2 as a common factor, we start by squaring both sides of (1) and get  $2 = n^2/d^2$ , so

$$2d^2 = n^2. \tag{2}$$

So 2 is a factor of  $n^2$ , which is only possible if 2 is in fact a factor of  $n$ .

This means that  $n = 2k$  for some integer  $k$  so

$$n^2 = (2k)^2 = 4k^2. \tag{3}$$

Combining (2) and (3) gives  $2d^2 = 4k^2$ , so

$$d^2 = 2k^2. \tag{4}$$

So 2 is a factor of  $d^2$ , which again is only possible if 2 is in fact also a factor of  $d$ , as claimed. ■

<sup>1</sup>Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

**Problem 3.**

Prove by cases that

$$\max(r, s) + \min(r, s) = r + s \quad (*)$$

for all real numbers  $r, s$ .

**Problem 4.**

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering  $\sqrt{2}^{\sqrt{2}}$  and arguing by cases.

**Problem 5.**

The fact that there are irrational numbers  $a, b$  such that  $a^b$  is rational was proved earlier by cases. Unfortunately, that proof was *nonconstructive*: it didn't reveal a specific pair  $a, b$  with this property. But in fact, it's easy to do this: let  $a ::= \sqrt{2}$  and  $b ::= 2 \log_2 3$ .

We know  $a = \sqrt{2}$  is irrational, and  $a^b = 3$  by definition. Finish the proof that these values for  $a, b$  work, by showing that  $2 \log_2 3$  is irrational.