In-Class Problems Week 14, Mon.

Problem 1.
There is an unpleasant, degenerative disease called Beaver Fever which causes people to tell math jokes unrelentingly in social settings, believing other people will think they're funny. Fortunately, Beaver Fever is rare, afflicting only about 1 in 1000 people. Doctor Abel has a fairly reliable diagnostic test to determine who is going to suffer from this disease:

- If a person will suffer from Beaver Fever, the probability that Dr. Abel diagnoses this is 0.99.
- If a person will not suffer from Beaver Fever, the probability that Dr. Abel diagnoses this is 0.97.

Let B be the event that a randomly chosen person will suffer **B**eaver Fever, and Y be the event that Dr. Abel's diagnosis is "**Y**es, this person will suffer from Beaver Fever," with $\overline{B}$ and $\overline{Y}$ being the complements of these events.

(a) The description above explicitly gives the values of the following quantities. What are their values?

$$\Pr[B] \qquad \Pr\left[Y \mid B\right] \qquad \Pr\left[\overline{Y} \mid \overline{B}\right]$$

(b) Write formulas for $\Pr[\overline{B}]$ and $\Pr\left[Y \mid \overline{B}\right]$ solely in terms of the explicitly given quantities in part (a)—literally use their expressions, not their numeric values.

(c) Write a formula for the probability that Dr. Abel says a person will suffer from Beaver Fever solely in terms of $\Pr[B]$, $\Pr[\overline{B}]$, $\Pr\left[Y \mid B\right]$ and $\Pr\left[Y \mid \overline{B}\right]$.

(d) Write a formula solely in terms of the expressions given in part (a) for the probability that a person will suffer Beaver Fever given that Doctor Abel says they will.

Suppose there was a vaccine to prevent Beaver Fever, but the vaccine was expensive or slightly risky itself. If you were sure you were going to suffer from Beaver Fever, getting vaccinated would be worthwhile, but even if Dr. Abel diagnosed you as a future sufferer of Beaver Fever, the probability you actually will suffer Beaver Fever remains low (about 1/32 by part (d)).

In this case, you might sensibly decide not to be vaccinated—after all, Beaver Fever is not *that* bad an affliction. So the diagnostic test serves no purpose in your case. You may as well not have bothered to get diagnosed. Even so, the test may be useful:

(e) Suppose Dr. Abel had enough vaccine to treat 2% of the population. If he randomly chose people to vaccinate, he could expect to vaccinate only 2% of the people who needed it. But by testing everyone and only vaccinating those diagnosed as future sufferers, he can expect to vaccinate a much larger fraction people who were going to suffer from Beaver Fever. Estimate this fraction.

Problem 2.
There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort,

the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). If the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), he names one of the two with equal probability.

Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that knowing what the guards says will reduce his chances, so he is better off not knowing. For example, if the guard says, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $1/2$ because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Dark Lord Sauron has made a typical mistake when reasoning about conditional probability. Using a tree diagram and the four-step method, **explain his mistake**. What is the probability that Sauron is released given that the guard says Foo-Foo is released?

Hint: Define the events S, F and "F" as follows:

$$\begin{aligned} \text{“}F\text{”} &= \text{Guard says Foo-Foo is released} \\ F &= \text{Foo-Foo is released} \\ S &= \text{Sauron is released} \end{aligned}$$

Problem 3.
There are two decks of cards, the red deck and the blue deck. They differ slightly in a way that makes drawing the eight of hearts slightly more likely from the red deck than from the blue deck.

One of the decks is randomly chosen and hidden in a box. You reach in the box and randomly pick a card that turns out to be the eight of hearts. You believe intuitively that this makes the red deck more likely to be in the box than the blue deck.

Your intuitive judgment about the red deck can be formalized and verified using some inequalities between probabilities and conditional probabilities involving the events

$$\begin{aligned} R &::= \textbf{R}\text{ed deck is in the box,} \\ B &::= \textbf{B}\text{lue deck is in the box,} \\ E &::= \textbf{E}\text{ight of hearts is picked from the deck in the box.} \end{aligned}$$

(a) State an inequality between probabilities and/or conditional probabilities that formalizes the assertion, "picking the eight of hearts from the red deck is more likely than from the blue deck."

(b) State a similar inequality that formalizes the assertion "picking the eight of hearts from the deck in the box makes the red deck more likely to be in the box than the blue deck."

(c) Assuming that initially each deck is equally likely to be the one in the box, prove that the inequality of part (a) implies the inequality of part (b).

(d) Suppose you couldn't be sure that the red deck and blue deck initially were equally likely to be in the box. Could you still conclude that picking the eight of hearts from the deck in the box makes the red deck more likely to be in the box than the blue deck? Briefly explain.

Supplemental problems

Problem 4.
Suppose you repeatedly flip a fair coin until you see the sequence `HTT` or `HHT`. What is the probability you see the sequence `HTT` first?

Hint: Try to find the probability that `HHT` comes before `HTT` conditioning on whether you first toss an `H` or a `T`. The answer is not $1/2$.

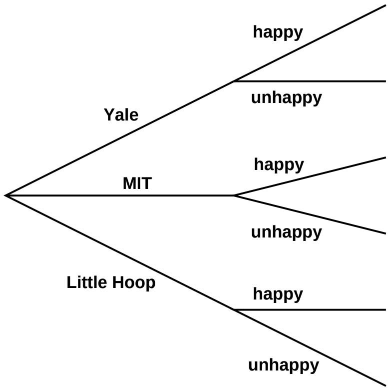
Problem 5.
Sally Smart just graduated from high school. She was accepted to three reputable colleges.

- With probability $4/12$, she attends Yale.
- With probability $5/12$, she attends MIT.
- With probability $3/12$, she attends Little Hoop Community College.

Sally is either happy or unhappy in college.

- If she attends Yale, she is happy with probability $4/12$.
- If she attends MIT, she is happy with probability $7/12$.
- If she attends Little Hoop, she is happy with probability $11/12$.

(a) A tree diagram to help Sally project her chance at happiness is shown below. On the diagram, fill in the edge probabilities, and at each leaf write the probability of the corresponding outcome.



(b) What is the probability that Sally is happy in college?

(c) What is the probability that Sally attends Yale, given that she is happy in college?

(d) Show that the event that Sally attends Yale **is not** independent of the event that she is happy.

(e) Show that the event that Sally attends MIT **is** independent of the event that she is happy.