In-Class Problems Week 12, Wed.

Problem 1.
Recall that for functions \( f, g \) on \( \mathbb{N} \), \( f = O(g) \) iff
\[
\exists c \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \quad (1)
\]

For each pair of functions below, determine whether \( f = O(g) \) and whether \( g = O(f) \). In cases where one function is \( O() \) of the other, indicate the smallest nonnegative integer \( c \) and for that smallest \( c \), the smallest corresponding nonnegative integer \( n_0 \) ensuring that condition (1) applies.

(a) \( f(n) = n^2, g(n) = 3n \).

(b) \( f(n) = (3n - 7)/(n + 4), g(n) = 4 \)

(c) \( f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n \)

Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

(i) \( f = o(g) \), the “little Oh” relation.

(ii) \( f = O(g) \), the “big Oh” relation.

(iii) \( f \sim g \), the “asymptotically equal” relation.

(iv) \( f = \Theta(g) \), the “Theta” relation.

(v) \( f = O(g) \) AND NOT \( g = O(f) \).

(b) Indicate the implications among the items (i)–(v) in part (a). For example,

- item (i) IMPLIES item (ii).

Briefly explain your answers.
Problem 3.

False Claim.

\[ 2^n = O(1). \]  \hspace{1cm} (2)

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

*Bogus proof.* The proof is by induction on \( n \) where the induction hypothesis \( P(n) \) is the assertion (2).

- **Base case:** \( P(0) \) holds trivially.
- **Inductive step:** We may assume \( P(n) \), so there is a constant \( c > 0 \) such that \( 2^n \leq c \cdot 1 \). Therefore,

\[
2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,
\]

which implies that \( 2^{n+1} = O(1) \). That is, \( P(n + 1) \) holds, which completes the proof of the inductive step.

We conclude by induction that \( 2^n = O(1) \) for all \( n \). That is, the exponential function is bounded by a constant.

\[ \blacksquare \]

**Supplemental problems**

Problem 4.
Assign true or false for each statement and prove it.

- \( n^2 \sim n^2 + n \)
- \( 3^n = O(2^n) \)
- \( n^{\sin(n\pi/2)+1} = o(n^2) \)
- \( n = \Theta\left(\frac{3n^3}{(n+1)(n-1)}\right) \)

Problem 5.
Give an elementary proof (without appealing to Stirling’s formula) that \( \log(n!) = \Theta(n \log n) \).