

In-Class Problems Week 12, Mon.

Problem 1.

We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour $1/3$ of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour $1/3$ of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

- (a) Describe a closed-form formula for the amount of wine in the first glass after n back-and-forth pourings.
- (b) What is the limit of the amount of wine in each glass as n approaches infinity?

Problem 2.

An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

- (a) What is the most distant point that the explorer can reach and then return to the oasis, with no water precached in the desert, if she takes a total of only 1 gallon from the oasis?
- (b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.
- (c) The explorer will travel using a recursive strategy to go far into the desert and back, drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n - 1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n - 1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the n th Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

- (d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Problem 3.

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a weakly decreasing function. Define

$$S ::= \sum_{i=1}^n f(i)$$

and

$$I ::= \int_1^n f(x) dx.$$

Prove that

$$I + f(n) \leq S \leq I + f(1).$$

(Proof by very clear picture is OK.)

Problem 4.

Sammy the Shark is a financial service provider who offers loans on the following terms.

- Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.
- Each evening, Sammy first charges a service fee which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p . For example, Sammy might charge a "modest" ten cent service fee and 1% interest rate per day, and then f would be 0.1 and p would be 1.01.

- What is the client's debt at the end of the first day?
- What is the client's debt at the end of the second day?
- Write a formula for the client's debt after d days and find an equivalent closed form.
- If you borrowed \$10 from Sammy for a year, how much would you owe him?

Supplemental problem**Problem 5.**

You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \dots + z^n \\ zS &= z + z^2 + \dots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \quad (\text{where } z \neq 1) \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \dots + nz^n$$