**Connected vertices**

**Lemma:**
The shortest walk between two vertices is a path!

**Proof:** (by contradiction) suppose path from u to v crossed itself:

\[ u \rightarrow c \rightarrow c \rightarrow v \]

then path without c---c is shorter!

**Walks & Paths**

**Lemma:**
The shortest walk between two vertices is a path!

**Proof:** (by contradiction) suppose path from u to v crossed itself:

\[ u \rightarrow c \rightarrow c \rightarrow v \]

then path without c---c is shorter!

**length n walk relation**

\[ v \xrightarrow{G^n} w \]

IFF \( \exists \) length n walk from v to w

\( G^n \) is the length n walk relation for \( G \)
length n walk relation

G itself is the length 1 walk relation: \( G^1 = G \)

lemma:

\[ G^m \circ G^n = G^{m+n} \]

relational composition

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Matrices & Composition

\( A_G \) := Adjacency matrix for \( G \)

Lemma:

\[ A_{G \circ H} = A_H \circ A_G \]

where \( \circ \) is Boolean matrix product—using OR instead of +

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Matrices & Composition

So compute \( A_G^n \) by fast matrix exponentiation

\[ \approx \log n \text{ matrix products.} \]
Walk Relation

\[ G^* \text{ is walk relation of } G \]
\[ u \text{ } G^* \text{ } v \text{ iff } \exists \text{walk } u \text{ to } v \]
(u is connected to v)

Compute the Walk Relation

Add self-loops:

\[ G^\leq := G \cup \text{Id}_V \]

\[ G^\leq \text{ has a length } n \text{ walk iff } G \text{ has a length } \leq n \text{ walk} \]

lengthening a walk in \( G \)

\[ \text{lengthening a walk in } G^\leq \]

just keep looping \( k \) times to make a length \( 5+k \) walk in \( G^\leq \)
Compute the Walk Relation

If $G$ has $n$ vertices, then
length of paths is $< n$, and
$G^* = \left( G^{\leq} \right)^{n-1}$

So find all connected vertex pairs with $n^2 \log n$ AND/OR's