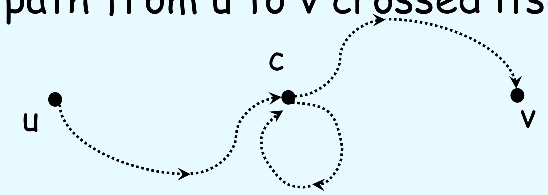
Mathematics for Computer Science
MIT 6.042J/18.062J

Connected vertices

Walks & Paths

Lemma:
The shortest walk between two vertices is a path!

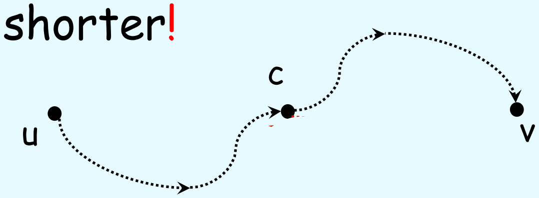
Proof: (by contradiction) suppose path from u to v crossed itself:



Walks & Paths

Lemma:
The shortest walk between two vertices is a path!

then path without c---c is shorter!



length n walk relation

v G^n w

IFF ∃ length n walk from v to w

G^n is the length n walk relation for G

length n walk relation

G itself is the length 1 walk relation: $G^1 = G$

lemma:

$$G^m \circ G^n = G^{m+n}$$

($\circ$: relational composition)

$G^m \circ G^n = G^{m+n}$

$x(G^m \circ G^n)y$ ::= $\exists z.\ x\ G^m\ z\ G^n\ y$

IFF $x\ G^{m+n}\ y$

because a length m+n walk consists of a length m walk followed by a length n walk

Matrices & Composition

A_G ::= Adjacency matrix for G

Lemma: $A_{G \circ H} = A_H \odot A_G$

where $\odot$ is Boolean matrix product—using OR instead of $+$

Matrices & Composition

So compute A_{G^n} by fast *matrix* exponentiation $\approx$ log n matrix products.

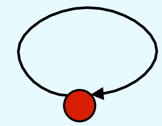
Walk Relation

G^* is walk relation of G

$u \; G^* \; v$ iff $\exists$walk u to v

(u is connected to v)

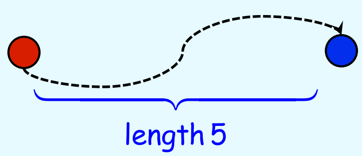
Compute the Walk Relation

Add self-loops: 

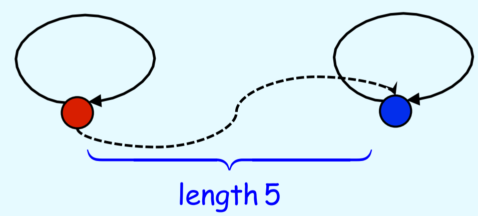
$$G^{\le} ::= G \cup Id_V$$

$G^{\le}$ has a length n walk iff G has a length $\le n$ walk

lengthening a walk in G



lengthening a walk in $G^{\le}$



just keep looping k times to make a length $5+k$ walk in $G^{\le}$

Compute the Walk Relation

If G has n vertices, then

length of paths is $< n$, and

$$G^* = \left(G^{\leq}\right)^{n-1}$$

So find all connected vertex pairs with $n^2 \log n$ AND/OR's