

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science  
MIT 6.042J/18.062J

# Connected vertices



Albert R Meyer March 15, 2013

connected.1

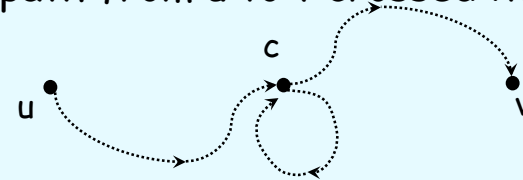
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## Walks & Paths

Lemma:

The **shortest** walk between  
two vertices is a path!

Proof: (by contradiction) suppose  
path from  $u$  to  $v$  crossed itself:



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connected.2

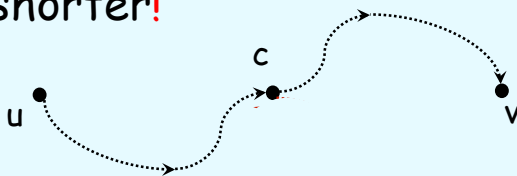
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## Walks & Paths

Lemma:

The **shortest** walk between  
two vertices is a path!

then path without  $c \text{---} c$  is  
shorter!



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connected.3

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## length $n$ walk relation

$$v \in G^n w$$

IFF  $\exists$  length  $n$  walk  
from  $v$  to  $w$

$G^n$  is the length  $n$   
walk relation for  $G$



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## length $n$ walk relation

$G$  itself is the length 1 walk relation:  $G^1 = G$

lemma:

$$G^m \circ G^n = G^{m+n}$$

relational composition



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$$G^m \circ G^n = G^{m+n}$$

$$x(G^m \circ G^n)y ::= \exists z. x G^m z G^n y$$

IFF  $x G^{m+n} y$

because a length  $m+n$  walk consists of a length  $m$  walk followed by a length  $n$  walk



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connected.6

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## Matrices & Composition

$A_G ::=$  Adjacency matrix for  $G$

Lemma:  $A_{G \circ H} = A_H \odot A_G$

where  $\odot$  is Boolean matrix product—using OR instead of +



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connected.8

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## Matrices & Composition

So compute  $A_{G^n}$  by fast matrix exponentiation  $\approx \log n$  matrix products.



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connected.9

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## Walk Relation

$G^*$  is walk relation of  $G$   
 $u G^* v$  iff  $\exists$  walk  $u$  to  $v$   
 ( $u$  is connected to  $v$ )



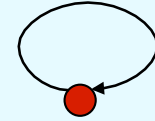
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connected.10

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## Compute the Walk Relation

Add self-loops:



$$G^{\leq} ::= G \cup \text{Id}_V$$

$G^{\leq}$  has a length  $n$  walk iff  
 $G$  has a length  $\leq n$  walk

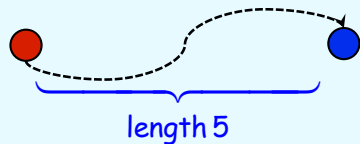


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connected.11

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## lengthening a walk in $G$

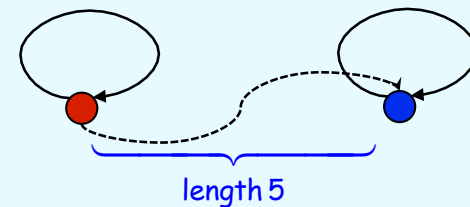


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connected.12

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## lengthening a walk in $G^{\leq}$



just keep looping  $k$  times to  
 make a length  $5+k$  walk in  $G^{\leq}$



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connected.13

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## Compute the Walk Relation

If  $G$  has  $n$  vertices, then  
length of paths is  $< n$ , and

$$G^* = \left( G^{\leq} \right)^{n-1}$$

So find **all connected vertex pairs** with  $n^2 \log n$  AND/OR's

